BIRZEIT UNIVERSITY

Faculty Of Graduate Studies
Masters program of Mathematics

Dynamics, Symmetry and Forbidden Sets of Some Difference Equations

By:
Kathem Badran

Supervisor:
Dr.Marwan Aloqeili

M.Sc.Thesis

May 4,2019

BIRZEIT UNIVERSITY

Faculty Of Graduate Studies
Masters program of MATHEMATICS

Dynamics, Symmetry and Forbidden Sets of Some Difference Equations

---

By:

KATHEM BADRAN

Supervisor:

DR.MARWAN ALOQEILI

" This Thesis was submitted in partial fulfillment of the requirements for the Master's Degree in Mathematics from the Faculty of Graduate studies at Birzeit University, Palestine."

May 4,2019

Faculty Of Graduate Studies
Masters program of Mathematics

Dynamics, Symmetry and Forbidden Sets of Some Difference Equations

by:
Kathem Badran

This thesis was successfully defended and approved on May 4,2019

Committee member:

Dr.Marwan Aloqeili(Supervisor) ____________

Dr.Alaeddin Elayyan(Member) ____________

Dr.Ala Talahmeh(Member) ____________

Acknowledgment

Foremost, I would express my gratitude due to the fact that I am getting close enough to achieve my dream. Thanks to God in the first place, I am expected to hold an MA successfully. So, thanks God for standing by my side. Also, all thanks are dedicated to all people who have supported me all through this way.

I can never forget my supervisor, Marwan AL-Oqaile, whom I always learn from his knowledge and patience that makes any work succeed unquestionably. Then, faculty members in the Department of Mathematics at Birzeit University do hold a big place in my heart that forgetting them is impossible. Last but not least, this work would not have been done without my great reference, which is represented in the thesis committee Dr. Alaeddin Elayyan and Dr. Ala Talahmeh with whom I have been caused and directed to useful and helpful comments on the thesis.

I would like to thank my school teachers for continous support until today. I dedicate all gratitude and respect to my classmates in Birzeit university (especially Wala' and Jameel, I thank you for your valuable observations and LaTex hints).

Finally, appreciation and love are presented to my family who has insisted on supporting me since the childhood.

Abstract

We study symmetry method to solve difference equation (18) in [3],

$$u_{n+2(i+1)} = \frac{u_n}{a + bu_{n+i+1}u_n},$$

by determining Lie groups of symmetries for even i. We find the forbidden set after we get an exact solution. We study also the local stability of the equilibrium point of this difference equation.

الملخص

سوف نقوم باستخدام طريقة تماثل لي لإياجد حل لمعادلة فرق من الدرجات العليا، ثم نستخدم هذا الحل لإيجاد المجموعات المنوعة، وندرس موضوع الثبات لنقاط التوازنن للمعادلة.

CONTENTS

INTRODUCTION

In the past thirty years, various types of difference equations have been considered and examined. These types of equations are of great importance in various fields of applied sciences. Deriving of the forbidden sets of the difference equations is a very important tool in studying the behaviour of the solutions of these difference equations.

Balbirea and Cascales [3] gave an explicit understanding of the forbidden sets by reviewing some of the previous studies. They proposed a list of open problems concerning this field.

Rabago [11] solved open problem 3, Eq. 17 in [3]. He found the forbidden set of the difference equation

$$u_{n+k+1} = \frac{u_{n+k}u_n}{au_{n+1} + u_{n+k}u_{n+1}u_n}$$

Abo-zeid [1] gave an exact solution of the difference equations

$$u_{n+3} = \frac{u_{n+2}u_n}{au_{n+2} + bu_n},$$

$$u_{n+3} = \frac{u_{n+1}u_n}{au_{n+1} + bu_n}$$

and then he determined the forbidden sets of them.

In this work, we are interested in finding a closed form solution of a certain class of difference equations using the method of symmetry. To be more precise, we are interested in addressing the solution to one of the open problems posted by Balibrea and Cascales in ([3], Open Problem 3, Eq. 18) concerning the forbidden set of a certain class of rational difference equations. Forbidden set is the set of initial conditions for which after a finite number of iterates we reach a value outside the domain of definition of the iteration function.

The first using of symmetry method was by Sophus Lie in order to solve ordinary differential equations (ODEs). For an introduction to symmetry methods for ODEs, see Hydon (2000). After that the method of symmetry was developed to solve ordinary difference equations by many researchers. Hydon (2000) introduced a method for obtaining the Lie symmetries and used it to reduce the order of the ordinary difference equations and to find the solution.

In this thesis, we devise topics as follows:

In chapter **1**, we introduce what do we mean about forbidden set concerning difference equations, and mention as an example recent results of determining the forbidden sets for some difference equations.

In chapter **2**, we investigate symmetries for first and second order difference equations, and we show how can we use symmetry to solve these equations. We generalize the symmetry method for higher order difference equations.

In chapter **3** and the aim of this thesis, we use symmetry method to solve difference equation

$$u_{n+2(i+1)} = \frac{u_n}{a + bu_{n+i+1}u_n},$$

for even i, with real initial conditions $\{u_n\}_{n=0}^{2i+1}$. We shall determine the forbidden set of this equation after we get closed form solution.

1. FORBIDDEN SET

In this chapter, we give an explicit definition of forbidden set and determine the forbidden set of some difference equations considered in recent papers, equation (1) from [12] and equation (1) from [3].

1.1 Introduction

Let k be a positive integer, a difference equation of order k is an equation of the form

$$u_{n+k} = f(u_{n+k-1}, \dots u_n), \tag{1.1}$$

where $f : A \subset \mathbb{R}^k \to \mathbb{R}$ is a continuous function.

Definition 1. *[3]* The *forbidden set* ($\mathcal{F}$) of difference equation (1.1) is the set of initial conditions for which after a finite number of iterates we reach a value outside the domain of definition of the iteration function, i.e.

$$\begin{aligned}\mathcal{F} = \{(u_{k-1}, \dots, u_0) : \exists\ n \geq 0 \mid f(u_{n+k-1}, \dots u_n) \text{ is not defined or}\\ \big(f(u_{n+k-1}, \dots u_n), u_{n+k-1}, \dots, u_n\big) \notin A\}.\end{aligned}$$

If we consider, for example, the simple difference equation

$$u_{n+1} = c, \qquad \text{where } c \in \mathbb{R},$$

then the forbidden set of this equation is $\mathcal{F} = \phi$ because $u_{n+1} = c$ is defined for all $n \geq 0$ where $c \in \mathbb{R}$.

Now, take

$$u_{n+1} = \frac{1}{u_n},$$

let $u_0 \neq 0$, then the sequence

$$u_0, u_1, u_2, u_3, u_4, u_5 \dots$$

under iteration function

$$u_{n+1} = f(u_n) = \frac{1}{u_n},$$

become

$$u_0, \frac{1}{u_0}, u_0, \frac{1}{u_0}, u_0, \frac{1}{u_0}, \ldots$$

which implies, every solution where $u_0 \neq 0$ is 2-periodic. On the other hand, if $u_0 = 0$, then u_1 is undefined, therefore, the forbidden set is $\mathcal{F} = \{u_0 = 0\}$.

The difference equation

$$u_{n+1} = \frac{1}{u_n^2 + 1},$$

has empty forbidden set ($\mathcal{F} = \phi$) in $\mathbb{R}$. Note that $i \in \mathcal{F}$ when this equation is taken over the complex field.

Example 1. [2]Determine the forbidden set of the following difference equations:

(a)

$$u_{n+1} = 2 - \frac{1}{u_n}. \tag{1.2}$$

(b)

$$u_{n+2} = \frac{u_n}{1 + u_{n+1}u_n}. \tag{1.3}$$

Solution. (a) $u_{n+1} = 2 - \frac{1}{u_n}$, calculate the terms of this sequence:

$$u_1 = 2 - \frac{1}{u_0} = \frac{2u_0 - 1}{u_0},$$

u_1 is undefined when $u_0 = 0$, hence $\mathcal{F}_1 = \{u_0 = 0\}$.

$$u_2 = 2 - \frac{1}{u_1} = 2 - \frac{u_0}{2u_0 - 1} = \frac{3u_0 - 2}{2u_0 - 1},$$

u_2 is undefined when $2u_0 - 1 = 0$, or $u_0 = \frac{1}{2}$, hence $\mathcal{F}_2 = \{u_0 = \frac{1}{2}\}$.

$$u_3 = 2 - \frac{1}{u_2} = 2 - \frac{2u_0 - 1}{3u_0 - 2} = \frac{4u_0 - 3}{3u_0 - 2},$$

u_3 is undefined when $3u_0 - 2 = 0$, or $u_0 = \frac{2}{3}$, hence $\mathcal{F}_3 = \{u_0 = \frac{2}{3}\}$.

$$u_4 = 2 - \frac{1}{u_3} = 2 - \frac{3u_0 - 2}{4u_0 - 3} = \frac{5u_0 - 4}{4u_0 - 3},$$

u_4 is undefined when $4u_0 - 3 = 0$, or $u_0 = \frac{3}{4}$, hence $\mathcal{F}_4 = \{u_0 = \frac{3}{4}\}$.
which implies,

$$\begin{aligned}\mathcal{F}_5 &= \{u_0 = \frac{4}{5}\}\\ \mathcal{F}_6 &= \{u_0 = \frac{5}{6}\}\\ &\vdots\end{aligned}$$

Thus, the forbidden set of equation (1.2) is

$$\begin{aligned}\mathcal{F} &= \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4 \cup \dots\\ &= \Big\{u_0 = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}\Big\}\\ &= \bigcup_{n\geq 0} \{u_0 : u_0 = \frac{n}{n+1}\}\end{aligned}$$

We can find this forbidden set through two ways, one of them by backward orbits and the other by solving the difference equation:

First way. Backward orbits:

Let

$$u_{n+1} = f(u_n) = 2 - \frac{1}{u_n},$$

so

$$u_{n-1} = f^{-1}(u_n) = \frac{1}{2 - u_n}$$

and so

$$u_0 = f^{-1}(u_1) = f^{-1}f^{-1}(u_2) = \cdots = f^{-n}(u_n)$$

then,

u_1 is undefined when $u_0 = 0$,

u_2 is undefined when $u_1 = 0$ so $u_0 = f^{-1}(u_1) = f^{-1}(0) = \dfrac{1}{2} \in \mathcal{F}$,

u_3 is undefined when $u_2 = 0$ so $u_0 = f^{-2}(u_2) = f^{-2}(0) = \dfrac{2}{3} \in \mathcal{F}$,

$\vdots$

u_{n+1} is undefined when $u_n = 0$ so $u_0 = f^{-n}(u_n) = f^{-n}(0) = \dfrac{n}{n+1} \in \mathcal{F}$.

Which implies

$$\mathcal{F} = \bigcup_{n\geq 0} \{u_0 : u_0 = \frac{n}{n+1}\}.$$

Now, we can deduce that, to find the forbidden set of equation (1.2), let

$$f(u_n) = u_{n+1} = 2 - \frac{1}{u_n}, \qquad f(u_n) \text{ is undefined when } u_n = 0,$$

then

$$f^{-1}(u_n) = \frac{1}{2-u_n}$$

Assume a new orbit s_k, is the backward orbit. Let

$$s_{k+1} = f^{-1}(s_k), \qquad s_0 = 0, \qquad k \geq 0,$$

this implies,

$$s_k = f^{-k}(s_0) = f^{-k}(0) = \frac{k}{k+1},$$

we start with initial value $s_0 = 0$ since $f(u_n)$ is undefined when $u_n = 0$, and for every term we moved in the backward orbit, we stay in $\mathcal{F}$. It follows that

$$\mathcal{F} = \bigcup_{n\geq 0} \{u_0 : u_0 = \frac{n}{n+1}\}$$

Second way: by solving the difference equation

$$u_{n+1} = 2 - \frac{1}{u_n}$$

let

$$u_n = \frac{v_{n+1}}{v_n}, \qquad v_n \neq 0, \quad n = 0, 1, 2, \ldots, \tag{1.4}$$

this substitution convert difference equation (1.2) to a second order linear homogeneous difference equation

$$v_{n+2} - 2v_{n+1} + v_n = 0, \tag{1.5}$$

the characteristic equation is

$$\lambda^2 - 2\lambda + 1 = 0,$$

so

$$(\lambda - 1)^2 = 0,$$

we have a repeated real root, $\lambda = 1$, then the solution of difference equation (1.5) is

$$\begin{aligned} v_n &= c_1(1)^n + c_2 n(1)^n \\ &= c_1 + c_2 n \end{aligned}$$

but

$$v_0 = c_1 \quad \text{and} \quad v_1 = c_1 + c_2,$$

so

$$c_2 = v_1 - v_0$$

which implies

$$v_n = v_0 + n(v_1 - v_0),$$

from substitution (1.4),

$$\begin{aligned} u_n &= \frac{v_{n+1}}{v_n} \\ &= \frac{v_0 + (n+1)(v_1 - v_0)}{v_0 + n(v_1 - v_0)} \end{aligned}$$

since $u_0 = \frac{v_1}{v_0}$ so $v_1 = u_0 v_0$. Thus

$$\begin{aligned} u_n &= \frac{v_0 + (n+1)(u_0 v_0 - v_0)}{v_0 + n(u_0 v_0 - v_0)} \\ &= \frac{1 + (n+1)(u_0 - 1)}{1 + n(u_0 - 1)}, \qquad \text{since } v_0 \neq 0 \\ &= \frac{u_0 n + u_0 - n}{1 + u_0 n - n}, \end{aligned}$$

u_n is undefined when

$$1 + u_0 n - n = 0.$$

Thus, forbidden sets $(\mathcal{F})$ of the difference equation (1.2) is

$$\begin{aligned} \mathcal{F} &= \bigcup_{n \geq 1} \{u_0 : u_0 = \frac{n-1}{n}\} \\ &= \bigcup_{n \geq 0} \{u_0 : u_0 = \frac{n}{n+1}\}. \end{aligned}$$

(b) $u_{n+2} = \frac{u_n}{1+u_{n+1}u_n}$, let u_0, u_1 be given such that $u_0u_1 \neq 0$, multiply both sides by u_{n+1} we get,

$$u_{n+2}u_{n+1} = \frac{u_{n+1}u_n}{1+u_{n+1}u_n}, \tag{1.6}$$

substitute $v_n = u_{n+1}u_n + 1$, this convert equation (1.6) into

$$v_{n+1} = 2 - \frac{1}{v_n}, \tag{1.7}$$

from previous example, the forbidden set of this equation is

$$\mathcal{F}_{(1)} = \bigcup_{n\geq 0} \{v_0 : v_0 = \frac{n}{n+1}\},$$

but $v_n = u_{n+1}u_n + 1$, so the forbidden set of the difference equation (1.3) is

$$\begin{aligned}
\mathcal{F} &= \bigcup_{n\geq 1} \{v_0 : v_0 = \frac{n-1}{n}\} \\
&= \bigcup_{n\geq 1} \{(u_1, u_0) : u_1u_0 + 1 = \frac{n-1}{n}\} \\
&= \bigcup_{n\geq 1} \{(u_1, u_0) : u_1u_0 = \frac{-1}{n}\}
\end{aligned}$$

1.2 Forbidden sets of some difference equations

Consider the difference equation

$$u_{n+2} = \frac{au_n}{u_{n+1}u_n + b},$$

substituting $u_n/\sqrt{b}$ for u_n and c for a/b gives the equivalent equation

$$u_{n+1} = \frac{cu_n}{u_{n+1}u_n + 1}. \tag{1.8}$$

Example 2. *[12]* The forbidden set $\mathcal{F}$ of difference equation (1.8) is a sequence of hyperbolas as follows:

$$\mathcal{F} = \bigcup_{n\geq 0} \{(u_1, u_0) : u_1u_0 = -\mu_n\}, \tag{1.9}$$

where

$$\mu_n = \begin{cases} \frac{c-1}{c^{n+1}-1}, & \text{if } c \neq 1; \\ \frac{1}{n+1}, & \text{if } c = 1. \end{cases}$$

Solution. *If $u_1u_0 \neq 0$, multiply both sides of equation (1.8) by u_{n+1}, we get*

$$u_{n+2}u_{n+1} = \frac{cu_{n+1}u_n}{u_{n+1}u_n + 1},$$

substitute

$$\frac{1}{v_n} = u_{n+1}u_n,$$

we obtain the first order difference equation

$$v_{n+1} = \frac{1}{c}v_n + \frac{1}{c}, \qquad v_0 = \frac{1}{u_1u_0},$$

let

$$f(v_n) = v_{n+1} = \frac{1}{c}v_n + \frac{1}{c}$$

then

$$f^{-1}(v_n) = cv_n - 1.$$

Assume a new orbit s_k is the backward orbit then $s_{k+1} = f^{-1}(s_k)$, $k \geq 0$, let the initial value $s_0 = -1$. Consider y_k such that $s_k = 1/y_{k+1}y_k$, so $y_1y_0 = -1$, then

$$\frac{1}{y_{k+1}y_k} = s_k = f^{-1}(s_{k-1}) = f^{-k}(s_0) = f^{-k}(-1) = -\sum_{i=0}^{k} c^i = -\begin{cases} \frac{c^{k+1}-1}{c-1}, & \text{if } c \neq 1; \\ k+1, & \text{if } c = 1. \end{cases}$$

In the backward orbit we start with initial point (y_1, y_0) with $y_1y_0 = -1$ and for every term we moved, we stay in $\mathcal{F}$. It follows that the points (y_{k+1}, y_k) are on hyperbolas(1.9), which completes the proof.

[1] Find the forbidden set of the difference equation

$$u_{n+3} = \frac{u_{n+2}u_n}{au_{n+2} + bu_n}, \qquad \text{where } a, b > 0. \tag{1.10}$$

Solution. We shall find the general solution of equation (1.10) and then we can determine the forbidden set. Let

$$u_n = \frac{1}{v_n},$$

this substitution converts equation (1.6) into the third order linear homogeneous difference equation

$$v_{n+3} - bv_{n+2} - av_n = 0, \tag{1.11}$$

the characteristic equation of this equation is

$$\lambda^3 - b\lambda^2 - a = 0, \tag{1.12}$$

the last equation has at least one real root since it's a polynomial of odd degree, say μ_1, so

$$\mu_1^3 - b\mu_1^2 - a = 0$$

then

$$\mu_1^3 = b\mu_1^2 + a \ > \ 0, \qquad \text{since } a, b > 0$$

which implies

$$\mu_1 > 0,$$

also,

$$\mu_1^3 > a \quad \text{and} \quad \mu_1^3 > b\mu_1^2,$$

therefore,

$$\mu_1 > \sqrt[3]{a} \quad \text{and} \quad \mu_1 > b, \qquad a, b, \mu_1 > 0.$$

Thus,

$$\mu_1 > \max\{\sqrt[3]{a}, b\}.$$

Now, to obtain other two roots $\mu_{2,3}$ of equation (1.12), divide equation(1.12) by $\lambda - \mu_1$, we get

$$\lambda^2 + (-b + \mu_1)\lambda + \mu_1^2 - \mu_1 b, \tag{1.13}$$

equation (1.12) can be written as

$$\begin{aligned} \lambda^3 - b\lambda^2 - a &= (\lambda - \mu_1)(\lambda^2 + (-b + \mu_1)\lambda + \mu_1^2 - \mu_1 b) \\ &= 0. \end{aligned}$$

The roots $\mu_{2,3}$ are the roots of the quadratic equation (1.13)

$$\begin{aligned} \mu_{2,3} &= \frac{-(\mu_1 - b) \pm \sqrt{(\mu_1 - b)^2 - 4(\mu_1^2 - b\mu_1)}}{2} \\ &= \frac{-(\mu_1 - b) \pm \sqrt{-3\mu_1^2 + 2b\mu_1 + b^2}}{2} \\ &= \frac{-(\mu_1 - b)}{2} \pm \frac{\sqrt{-(3\mu_1 + b)(\mu_1 - b)}}{2}, \end{aligned}$$

but $\mu_1 - b > 0$ since $\mu_1 > b > 0$, so

$$-(3\mu_1 + b)(\mu_1 - b) < 0,$$

it follows that

$$\begin{aligned}
\mu_{2,3} &= \frac{-(\mu_1 - b)}{2} \pm i\frac{\sqrt{(3\mu_1 + b)(\mu_1 - b)}}{2} \\
&= \frac{-(\mu_1 - b)}{2} \pm i\frac{\sqrt{3\mu_1^2 - 2\mu_1 b - b^2}}{2}
\end{aligned}$$

$$\begin{aligned}
|\mu_{2,3}| &= \sqrt{\Big(\frac{-(\mu_1 - b)}{2}\Big)^2 + \Big(\frac{\sqrt{3\mu_1^2 - 2\mu_1 b - b^2}}{2}\Big)^2} \\
&= \sqrt{\frac{\mu_1^2 - 2\mu_1 b + b^2}{4} + \frac{3\mu_1^2 - 2\mu_1 b - b^2}{4}} \\
&= \sqrt{\mu_1^2 - \mu_1 b} \\
&= \sqrt{\frac{a}{\mu_1}}.
\end{aligned}$$

Let

$$\begin{aligned}
\theta &= \tan^{-1}\Big(\frac{\big(\sqrt{(3\mu_1 + b)(\mu_1 - b)}\big)/2}{\big(-(\mu_1 - b)\big)/2}\Big) \\
&= \tan^{-1}\Bigg(-\sqrt{\frac{3\mu_1 + b}{\mu_1 - b}}\Bigg) \quad \in \ (\pi/2, \pi)
\end{aligned}$$

then

$$\begin{aligned}
\mu_2 &= |\mu_2|\, e^{i\theta} = \Big(\frac{a}{\mu_1}\Big)^{1/2} e^{i\theta} \\
\mu_3 &= |\mu_3|\, e^{-i\theta} = \Big(\frac{a}{\mu_1}\Big)^{1/2} e^{-i\theta}.
\end{aligned}$$

Hence, the solution of equation (1.11)

$$\begin{aligned}
v_n &= c_1\mu_1^n + \hat{c}_2\mu_2^n + \hat{c}_3\mu_3^n \\
&= c_1\mu_1^n + \hat{c}_2\Big(\big(\frac{a}{\mu_1}\big)^{1/2} e^{i\theta}\Big)^n + \hat{c}_3\Big(\big(\frac{a}{\mu_1}\big)^{1/2} e^{-i\theta}\Big)^n \\
&= c_1\mu_1^n + c_2\Big(\frac{a}{\mu_1}\Big)^{n/2}\cos n\theta + c_3\Big(\frac{a}{\mu_1}\Big)^{n/2}\sin n\theta,
\end{aligned}$$

where $c_1, \hat{c}_2, \hat{c}_3$ are constants and

$$c_2 = \hat{c}_2 + \hat{c}_3,\;\; c_3 = i(\hat{c}_2 - \hat{c}_3).$$

Let v_0, v_1, v_2 be given, then

$$\begin{aligned} v_0 &= c_1 + c_2, \\ v_1 &= c_1\mu_1 + c_2\left(\frac{a}{\mu_1}\right)^{1/2}\cos\theta + c_3\left(\frac{a}{\mu_1}\right)^{1/2}\sin\theta, \\ v_2 &= c_1\mu_1^2 + c_2\left(\frac{a}{\mu_1}\right)\cos 2\theta + c_3\left(\frac{a}{\mu_1}\right)\sin 2\theta, \end{aligned}$$

solving this system of equations for c_1, c_2, c_3, we get

$$c_1 = \frac{\Delta_1}{\Delta}, \quad c_2 = \frac{\Delta_2}{\Delta}, \quad c_3 = \frac{\Delta_3}{\Delta}$$

where

$$\Delta = \begin{vmatrix} 1 & 1 & 0 \\ \mu_1 & \left(\frac{a}{\mu_1}\right)^{1/2}\cos\theta & \left(\frac{a}{\mu_1}\right)^{1/2}\sin\theta \\ \mu_1^2 & \frac{a}{\mu_1}\cos 2\theta & \frac{a}{\mu_1}\sin 2\theta \end{vmatrix}$$

$$\Delta_1 = \begin{vmatrix} v_0 & 1 & 0 \\ v_1 & \left(\frac{a}{\mu_1}\right)^{1/2}\cos\theta & \left(\frac{a}{\mu_1}\right)^{1/2}\sin\theta \\ v_2 & \frac{a}{\mu_1}\cos 2\theta & \frac{a}{\mu_1}\sin 2\theta \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} 1 & v_0 & 0 \\ \mu_1 & v_1 & \left(\frac{a}{\mu_1}\right)^{1/2}\sin\theta \\ \mu_1^2 & v_2 & \frac{a}{\mu_1}\sin 2\theta \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & v_0 \\ \mu_1 & \left(\frac{a}{\mu_1}\right)^{1/2}\cos\theta & v_1 \\ \mu_1^2 & \frac{a}{\mu_1}\cos 2\theta & v_2 \end{vmatrix}$$

then,

$$\begin{aligned} c_1 &= \frac{\Delta_1}{\Delta} \\ &= \frac{1}{\Delta}\left(v_0\left[\left(\frac{a}{\mu_1}\right)^{3/2}(\cos\theta\sin 2\theta-\cos 2\theta\sin\theta)\right]+v_1\left[-\frac{a}{\mu_1}\sin 2\theta\right]+v_2\left[\left(\frac{a}{\mu_1}\right)^{1/2}\sin\theta\right]\right) \\ &= \frac{1}{\Delta}\left(\frac{1}{u_0}\left[\left(\frac{a}{\mu_1}\right)^{3/2}\sin\theta\right]+\frac{1}{u_0}\left[-\frac{a}{\mu_1}\sin 2\theta\right]+\frac{1}{u_2}\left[\left(\frac{a}{\mu_1}\right)^{1/2}\sin\theta\right]\right), \end{aligned}$$

$$\begin{aligned} c_2 &= \frac{\Delta_2}{\Delta} \\ &= \frac{1}{\Delta}\left(v_0\left[-a\sin 2\theta+\mu_1^{3/2}a^{1/2}\sin\theta\right]+v_1\left[\frac{a}{\mu_1}\sin 2\theta\right]+v_2\left[\left(\frac{a}{\mu_1}\right)^{1/2}\sin 2\theta\right]\right) \\ &= \frac{1}{\Delta}\left(\frac{1}{u_0}\left[a\sin 2\theta-\mu_1^{3/2}a^{1/2}\sin\theta\right]+\frac{1}{u_1}\left[\frac{a}{\mu_1}\sin 2\theta\right]+\frac{1}{u_2}\left[\left(\frac{a}{\mu_1}\right)^{1/2}\sin 2\theta\right]\right) \end{aligned}$$

$$\begin{aligned} c_3 &= \frac{\Delta_3}{\Delta} \\ &= \frac{1}{\Delta}\left(v_0\left[a\cos 2\theta-\mu_1^{3/2}a^{1/2}\cos\theta\right]+v_1\left[-\frac{a}{\mu_1}\cos 2\theta+\mu_1^2\right]+v_2\left[\left(\frac{a}{\mu_1}\right)^{1/2}\cos\theta-\mu_1\right]\right) \\ &= \frac{1}{\Delta}\left(\frac{1}{u_0}\left[a\cos 2\theta-\mu_1^{3/2}a^{1/2}\cos\theta\right]+\frac{1}{u_1}\left[-\frac{a}{\mu_1}\cos 2\theta+\mu_1^2\right]+\frac{1}{u_2}\left[\left(\frac{a}{\mu_1}\right)^{1/2}\cos\theta-\mu_1\right]\right). \end{aligned}$$

Thus, the solution of difference equation (1.10) is

$$u_n=\frac{1}{v_n}=\frac{1}{c_1\mu_1^n+c_2\left(\frac{a}{\mu_1}\right)^{n/2}\cos n\theta+c_3\left(\frac{a}{\mu_1}\right)^{n/2}\sin n\theta}.$$

The forbidden set of equation (1.10) is the set of all initial values for which u_n is undefined, then

$$\begin{aligned}
v_n &= 0\\
&= c_1\mu_1^n + c_2\Big(\frac{a}{\mu_1}\Big)^{n/2}\cos n\theta + c_3\Big(\frac{a}{\mu_1}\Big)^{n/2}\sin n\theta\\
&= \frac{1}{\Delta}\Bigg(\frac{1}{u_0}\Big[\Big(\frac{a}{\mu_1}\Big)^{3/2}\sin\theta\Big] + \frac{1}{u_1}\Big[-\frac{a}{\mu_1}\sin 2\theta\Big] + \frac{1}{u_2}\Big[\Big(\frac{a}{\mu_1}\Big)^{1/2}\sin\theta\Big]\Bigg)\mu_1^n\\
&\quad + \frac{1}{\Delta}\Bigg(\frac{1}{u_0}\Big[a\sin 2\theta - \mu_1^{3/2}a^{1/2}\sin\theta\Big] + \frac{1}{u_1}\Big[\frac{a}{\mu_1}\sin 2\theta\Big]\\
&\qquad\qquad + \frac{1}{u_2}\Big[\Big(\frac{a}{\mu_1}\Big)^{1/2}\sin 2\theta\Big]\Bigg)\Big(\frac{a}{\mu_1}\Big)^{n/2}\cos n\theta\\
&\quad + \frac{1}{\Delta}\Bigg(\frac{1}{u_0}\Big[a\cos 2\theta - \mu_1^{3/2}a^{1/2}\cos\theta\Big] + \frac{1}{u_1}\Big[-\frac{a}{\mu_1}\cos 2\theta + \mu_1^2\Big]\\
&\qquad\qquad + \frac{1}{u_2}\Big[\Big(\frac{a}{\mu_1}\Big)^{1/2}\cos\theta - \mu_1\Big]\Bigg)\Big(\frac{a}{\mu_1}\Big)^{n/2}\sin n\theta
\end{aligned}$$

which implies

$$\begin{aligned}
0 &= \frac{1}{u_0}\frac{1}{\Delta}\Bigg(\Big[\Big(\frac{a}{\mu_1}\Big)^{3/2}\sin\theta\Big]\mu_1^n + \Big[a\sin 2\theta - \mu_1^{3/2}a^{1/2}\sin\theta\Big]\Big(\frac{a}{\mu_1}\Big)^{n/2}\cos n\theta\\
&\qquad\qquad + \Big[a\cos 2\theta - \mu_1^{3/2}a^{1/2}\cos\theta\Big]\Big(\frac{a}{\mu_1}\Big)^{n/2}\sin n\theta\Bigg)\\
&\quad + \frac{1}{u_1}\frac{1}{\Delta}\Bigg(\Big[-\frac{a}{\mu_1}\sin 2\theta\Big]\mu_1^n + \Big[\frac{a}{\mu_1}\sin 2\theta\Big]\Big(\frac{a}{\mu_1}\Big)^{n/2}\cos n\theta\\
&\qquad\qquad + \Big[-\frac{a}{\mu_1}\cos 2\theta + \mu_1^2\Big]\Big(\frac{a}{\mu_1}\Big)^{n/2}\sin n\theta\Bigg)\\
&\quad + \frac{1}{u_2}\frac{1}{\Delta}\Bigg(\Big[\Big(\frac{a}{\mu_1}\Big)^{1/2}\sin\theta\Big]\mu_1^n + \Big[\Big(\frac{a}{\mu_1}\Big)^{1/2}\sin 2\theta\Big]\Big(\frac{a}{\mu_1}\Big)^{n/2}\cos n\theta\\
&\qquad\qquad + \Big[\Big(\frac{a}{\mu_1}\Big)^{1/2}\cos\theta - \mu_1\Big]\Big(\frac{a}{\mu_1}\Big)^{n/2}\sin n\theta\Bigg).
\end{aligned}$$

Thus, the forbidden set of difference equation (1.10)

$$\mathcal{F} = \bigcup_{n\geq 0}\Big\{(u_2, u_1, u_0) \in \mathbb{R}^3 : \frac{\alpha_{2n}}{u_2} + \frac{\alpha_{1n}}{u_1} + \frac{\alpha_{0n}}{u_0}\Big\}$$

where

$$\alpha_{2n} = \frac{1}{\Delta}\Bigg(\Big[\Big(\frac{a}{\mu_1}\Big)^{1/2}\sin\theta\Big]\mu_1^n + \Big[\Big(\frac{a}{\mu_1}\Big)^{1/2}\sin 2\theta\Big]\Big(\frac{a}{\mu_1}\Big)^{n/2}\cos n\theta + \Big[\Big(\frac{a}{\mu_1}\Big)^{1/2}\cos\theta - \mu_1\Big]\Big(\frac{a}{\mu_1}\Big)^{n/2}\sin n\theta\Bigg)$$

$$\alpha_{1n} = \frac{1}{\Delta}\Bigg(\Big[-\frac{a}{\mu_1}\sin 2\theta\Big]\mu_1^n + \Big[\frac{a}{\mu_1}\sin 2\theta\Big]\Big(\frac{a}{\mu_1}\Big)^{n/2}\cos n\theta + \Big[-\frac{a}{\mu_1}\cos 2\theta + \mu_1^2\Big]\Big(\frac{a}{\mu_1}\Big)^{n/2}\sin n\theta\Bigg)$$

$$\alpha_{0n} = \frac{1}{\Delta}\Bigg(\Big[\Big(\frac{a}{\mu_1}\Big)^{3/2}\sin\theta\Big]\mu_1^n + \Big[a\sin 2\theta - \mu_1^{3/2}a^{1/2}\sin\theta\Big]\Big(\frac{a}{\mu_1}\Big)^{n/2}\cos n\theta + \Big[a\cos 2\theta - \mu_1^{3/2}a^{1/2}\cos\theta\Big]\Big(\frac{a}{\mu_1}\Big)^{n/2}\sin n\theta\Bigg)$$

2. SYMMETRY METHOD

To clarify the concept of symmetries of an ordinary difference equation (OΔE), it is helpful to consider the symmetries of geometrical object. A symmetry of a geometrical object is an invertible transformation that maps the object to itself.

Consider the result of rotating a square anticlockwise about its centre. After a rotation of $\pi/2$, the square looks the same as it did before the rotation, so this transformation is a symmetry. Rotations of π, $3\pi/2$ and 2π are also symmetries of the square. In fact, rotating by 2π is equivalent to doing nothing, because each point is mapped to itself.

$Figure(1).\ Rotation\ of\ the\ square$

Definition 2. *[9]*(Trivial symmetry) The transformation mapping each point to it self.

In addition to the rotations described above, the reflections about the four axes marked in Figure(1) are also symmetries. So the square has eight distinct symmetries.

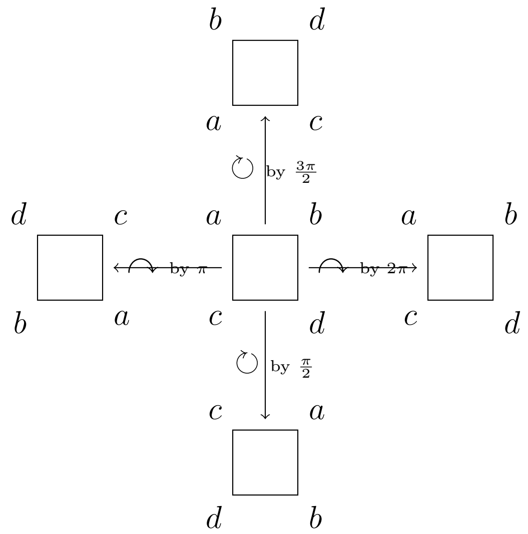
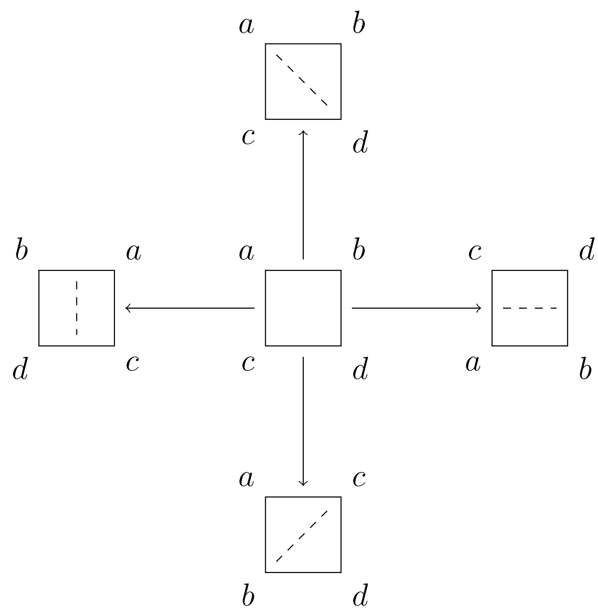


Fig. 2.1: dd



$Figure(2).\ Refliction\ of\ the\ square$

Definition 3. *[9]*A transformation is a symmetry if it satisfies the following:

(a) The transformation preserves the structure.

(b) The transformation is a diffeomorphism (a smooth invertible mapping whose inverse is also smooth).

(c) The transformation maps the object to itself.

Remark 1. *(i) Every object has at least one symmetry which is trivial symmetry.*

(ii) Each symmetry has a unique inverse, which is itself a symmetry.

(iii) The combined action of the symmetry and its inverse upon the object (in either order) leaves the object unchanged.

Theorem 2.0.1. *[10] The set of all symmetries of a geometrical object is a group.*

Example 3. *[10]* The group of symmetries of the square is called dihedral group D_4, with two generators Γ_1 and Γ_2 that are shown in Figure (3).

Γ_1 : rotation by $\pi/2$ (anticlockwise) about the square's centre.

Γ_2: reflection in a centreline.

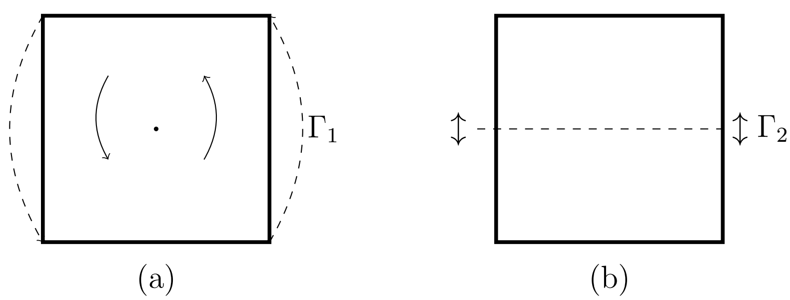


Figure (3). Some symmetries of a square:(a)rotation by $\frac{\pi}{2}$:(b)reflection.

These generators are subject to the relations

$$(\Gamma_1)^4 = (\Gamma_2)^2 = (\Gamma_1\Gamma_2)^2 = \text{identity map } (\Gamma_0)$$

We can see that $\Gamma_1,\ (\Gamma_1)^2,\ (\Gamma_1)^3$ and $(\Gamma_1)^4 = \Gamma_0$, represent the rotations by $\pi/2, \pi, 3\pi/2$ and 2π respectively, and $\Gamma_2,\ \Gamma_1\Gamma_2,\ (\Gamma_1)^2\Gamma_2$ and $(\Gamma_1)^3\Gamma_2$, represent the flips about the four axes.

So the eight distinct symmetries of the square are the elements of dihedral group,

$$D_4 = \{id, \Gamma_1,\ (\Gamma_1)^2,\ (\Gamma_1)^3, \Gamma_2,\ \Gamma_1\Gamma_2,\ (\Gamma_1)^2\Gamma_2,\ (\Gamma_1)^3\Gamma_2\}.$$

Note that $\Gamma_1\Gamma_2 \neq \Gamma_2\Gamma_1$.

2.1 Symmetries and Lie symmetries of difference equations

An OΔE (of any order) may be represented by the set of its solution. For an OΔE, symmetries are defined as follow.

Definition 4. *[10]* A symmetry of a given OΔE is a locally-defined diffeomorphism, Γ, that maps the set of all solutions to itself. (Consequently, every solution is mapped invertibly to a solution.)

Example 4. *[10]* For $\epsilon \in \mathbb{R}$, let

$$\Gamma_\epsilon : u \mapsto \hat{u} = e^\epsilon u,$$

be a transformation. We consider its affect on a linear homogeneous OΔE of order k. If $v_1, v_2, \ldots, v_k$ are linearly independent solutions, then the general solution is

$$u = \sum_{i=1}^{k} c_i v_i, \qquad c_i\text{'s are constants.}$$

The mapped solution, $\hat{u}$, is

$$\hat{u} = e^\epsilon \sum_{i=1}^{p} c_i v_i = \sum_{i=1}^{p} \hat{c}_i v_i, \qquad \text{where } \hat{c_i} = e^\epsilon c_i,$$

so every solution is mapped invertibly to a solution. Thus, Γ_ϵ is a symmetry of the OΔE for all $\epsilon \in \mathbb{R}$.

Definition 5. *[6]* A parametrized set of point transformations,

$$\Gamma_\epsilon : x \mapsto \hat{x}(x, \epsilon), \qquad \epsilon \in (\epsilon_0, \epsilon_1)$$

where $\epsilon_0 < 0 < \epsilon_1$, is a one parameter local Lie group if:

(L1) Γ_0 is the identity map, so that $\hat{x} = x$ when $\epsilon = 0$.

(L2) $\Gamma_\alpha\Gamma_\beta = \Gamma_{\alpha+\beta}$ for every α, β sufficiently close to zero.

(L3) Each $\hat{x}$ can be represented as a Taylor series in ϵ (in a neighbourhood of $\epsilon = 0$ that is determined by x), and therefore

$$\hat{x}(x, \epsilon) = x + \epsilon\eta(x) + O(\epsilon^2)$$

A local Lie group may not be group, unless it satisfies group axioms for sufficiently small parameter values. A one-parameter local Lie group of symmetries of a difference equation will depend on n and the continuous variable $u_n\big(i.e.\ \hat{u}_n = \hat{u}_n(n, u_n)\big)$. Since n is a discrete variable that cannot be changed by an arbitrarily small amount, so every one-parameter local Lie group of symmetries must leave n unchanged. We call symmetries that belong to a one-parameter local Lie group as "Lie symmetries".

Example 5. *[10]* The transformation

$$\Gamma_\epsilon : (n, u_n) \mapsto (\hat{n}, \hat{u}_n) = (n, e^\epsilon u_n), \qquad \epsilon \in \mathbb{R}$$

is a one-parameter local Lie group.

- (L1) When $\epsilon = 0$, we obtain $(\hat{n}, \hat{u}_n) = (n, u_n)$, so Γ_0 is the identity map.
- (L2) $\Gamma_\beta : (n, u_n) \mapsto (n, e^\beta u_n)$, which implies

  $$\Gamma_\alpha \Gamma_\beta : (n, e^\beta u_n) \mapsto (n, e^\alpha e^\beta u_n) = (n, e^{\alpha+\beta} u_n).$$

  Thus, $\Gamma_\alpha \Gamma_\beta = \Gamma_{\alpha+\beta}$.
- (L3) $\hat{u}_n$ can be represented as a Taylor series in ϵ

  $$\hat{u}_n = e^\epsilon u_n = \big(1 + \epsilon + o(\epsilon^2)\big) u_n = u_n + \epsilon u_n + O(\epsilon^2).$$

Now, consider this transformation affect on a linear homogeneous O∆E of order k, from Example 4, Γ_ϵ is a symmetry of this O∆E for every $\epsilon \in \mathbb{R}$. So Γ_ϵ is a Lie symmetry

Example 6. *[10]* consider the difference equation,

$$u_{n+1} - u_n = 0. \tag{2.1}$$

The transformation

$$\Gamma_\epsilon : (n, u_n) \mapsto (\hat{n}, \hat{u}_n) = (n, u_n + \epsilon), \qquad a \in \mathbb{R} \tag{2.2}$$

is a Lie symmetry, since Γ_ϵ is a symmetry for equation (2.1), and it is a one parameter local Lie group.

2.2 Characteristics and Canonical Coordinates

In this thesis we restrict attention to Lie symmetries for which $\hat{u}_n$ depends on n and u_n only. These are called Lie point symmetries, they are of the form

$$\hat{n} = n, \qquad \hat{u}_n = u_n + \epsilon Q(n, u_n) + O(\epsilon^2). \tag{2.3}$$

To see how such symmetries transform the shifted variables u_{n+k}, simply replace the free variable n in (2.3) by $n+k$:

$$\hat{u}_{n+k} = u_{n+k} + \epsilon Q(n+k, u_{n+k}) + O(\epsilon^2).$$

This is called the prolongation formula for Lie point symmetries.

The function $Q(n, u_n)$ is called the characteristic with respect to the coordinates (n, u_n). For instance, the characteristic that corresponds to the transformation (2.2),

$$\hat{n} = n, \qquad \hat{u}_n = u_n + \epsilon$$

is $Q(n, u_n) = 1$.

Consider changing of coordinates from (n, u_n) to (n, w_n), where $w'_n = \frac{\partial w_n}{\partial u_n} \neq 0$, then

$$\begin{aligned}
\hat{w}(n, u_n) &= w(\hat{n}, \hat{u}_n) \\
&= w(n, u_n + \epsilon Q(n, u_n) + O(\epsilon^2)) \qquad \text{by using relation (2.3)} \\
&= \Big(w\big(n, u_n + \epsilon Q(n, u_n)\big)\Big)_{|\epsilon=0} \\
&\quad + (\epsilon - 0)\left(\frac{d}{d\epsilon}\big(w(n, u_n + \epsilon Q(n, u_n))\big)\right)_{|\epsilon=0} + O(\epsilon^2), \text{ by applying Taylor's theorem} \\
&= w(n, u_n) + \epsilon\left(\left(\frac{d}{d\hat{u}}\big(w(n, u_n + \epsilon Q(n, u_n))\big)\right)\left(\frac{d\hat{u}}{d\epsilon}\right)\right)_{|\epsilon=0} + O(\epsilon^2) \\
&= w(n, u_n) + \epsilon w'(n, u_n) Q(n, u_n) + O(\epsilon^2) \\
&= w(n, u_n) + \epsilon \tilde{Q}(n, w_n) + O(\epsilon^2)
\end{aligned} \tag{2.4}$$

where,

$$\tilde{Q}(n, w_n) = w'(n, u_n) Q(n, u_n),$$

and it is called the characteristic with respect to (n, w_n).

Now, we introduce a canonical coordinate, s_n, such that

$$\left(\hat{n}, \hat{s}_n\right) = (n, s_n + \epsilon) \qquad \epsilon \in \mathbb{R}.$$

The characteristic with respect to (n, s_n) is $\tilde{Q}(n, s_n) = 1 = s'(n, u_n)Q(n, u_n)$, so

$$s(n, u_n) = \int \frac{1}{Q(n, u_n)}.$$

2.3 Linearized Symmetry Condition (LSC)

To determine the Lie point symmetries for a given difference equation, we find the characteristics by solving the LSC. This will be explained in this section.

2.3.1 First order difference equations

Given a first order difference equation,

$$u_{n+1} = \omega(n, u_n), \tag{2.5}$$

with a one-parameter local Lie group of symmetries, so the set of solutions of (2.5) is mapped to itself and

$$\hat{u}_{n+1} = \omega(\hat{n}, \hat{u}_n) \qquad \textit{when} \qquad u_{n+1} = \omega(n, u_n), \tag{2.6}$$

is satisfied which is called the symmetry condition.

Expand the symmetry condition (2.6) in powers of ϵ,

$$\begin{aligned} \hat{u}_{n+1}\Big|_{u_{n+1}=\omega(n,u_n)} &= \{u_{n+1} + \epsilon Q(n+1, u_{n+1}) + O(\epsilon^2)\}\Big|_{u_{n+1}=\omega(n,u_n)} \\ &= \omega(n, u_n) + \epsilon Q(n+1, \omega(n, u_n)) + O(\epsilon^2) \\ &= \omega(\hat{n}, \hat{u}_n), \qquad \textit{from } (2.6) \\ &= \omega(n, u_n) + \epsilon\omega'(n, u_n)Q(n, u_n) + O(\epsilon^2), \qquad \textit{by } (2.4) \end{aligned}$$

by comparing coefficients of ϵ, we obtain

$$Q(n+1, \omega(n, u_n)) = \omega'(n, u_n)Q(n, u_n). \tag{2.7}$$

This is called the linearized symmetry condition (LSC) for the difference equation (2.5).

The LSC (2.7) is a linear functional equation that may be difficult to solve completely. There is no real need to find the general solution of the LSC, as a single nonzero solution of the LSC is sufficient to determine the general solution of the O∆E. A practical approach is to use an ansatz (trial solution). Many physically important Lie point symmetries have characteristics of the form

$$Q(n, u_n) = \alpha_n u_n^2 + \beta_n u_n + \gamma_n \tag{2.8}$$

By substituting (2.8) into the LSC (2.7) and comparing powers of u_n, we obtain a linear system of O∆Es for the coefficients α_n, β_n and γ_n.

Example 7. *[10]* Find the characteristics of Lie point symmetries for

$$u_{n+1} = \frac{nu_n + 1}{u_n + n} = \omega(n, u_n), \qquad n \geq 2 \tag{2.9}$$

Solution.

$$\omega'(n, u_n) = \frac{\partial \omega(n, u_n)}{\partial u_n} = \frac{n^2 - 1}{(u_n + n)^2}$$

then the LSC for equation (2.9) is

$$Q\Big(n + 1, \frac{nu_n + 1}{u_n + n}\Big) = \frac{n^2 - 1}{(u_n + n)^2} Q(n, u_n),$$

with the ansatz (2.8), we get

$$\alpha_{n+1} u_{n+1}^2 + \beta_{n+1} u_{n+1} + \gamma_{n+1} = \tfrac{n^2-1}{(u_n+n)^2}\big(\alpha_n u_n^2 + \beta_n u_n + \gamma_n\big),$$

but $u_{n+1} = \frac{nu_n+1}{u_n+n}$, so

$$\alpha_{n+1}\Big(\frac{nu_n + 1}{u_n + n}\Big)^2 + \beta_{n+1}\frac{nu_n + 1}{u_n + n} + \gamma_{n+1} = \frac{n^2 - 1}{(u_n + n)^2}\big(\alpha_n u_n^2 + \beta_n u_n + \gamma_n\big),$$

multiplying by $(u_n + n)^2$, we obtain

$$\begin{aligned} n^2\alpha_{n+1}u_n^2 + 2n\alpha_{n+1}u_n + \alpha_{n+1} + n\beta_{n+1}u_n^2 + (n^2 + 1)\beta_{n+1}u_n + n\beta_{n+1} + \gamma_{n+1}u_n^2 + 2n\gamma_{n+1}u_n \\ + n^2\gamma_{n+1} \;=\; (n^2 - 1)\alpha_n u_n^2 + (n^2 - 1)\beta_n u_n + (n^2 - 1)\gamma_n. \end{aligned}$$

By comparing the powers of u, we get a system of difference equations:

$$u^2 \text{ terms}: \qquad n^2\alpha_{n+1} + n\beta_{n+1} + \gamma_{n+1} = (n^2 - 1)\alpha_n, \tag{2.10}$$

$$u \text{ term}: \qquad 2n\alpha_{n+1} + (n^2 + 1)\beta_{n+1} + 2n\gamma_{n+1}u_n = (n^2 - 1)\beta_n, \tag{2.11}$$

$$\text{other terms}: \qquad \alpha_{n+1} + n\beta_{n+1} + n^2\gamma_{n+1} = (n^2 - 1)\gamma_n, \tag{2.12}$$

subtracting (2.12) from (2.10), we get

$$\alpha_{n+1} - \gamma_{n+1} = \alpha_n - \gamma_n,$$

so

$$\alpha_n - \gamma_n = k_1, \qquad k_1 \text{ is a constant,}$$

adding (2.12) to (2.10), we get

$$(n^2+1)\alpha_{n+1} + 2n\beta_{n+1} + (n^2+1)\gamma_{n+1} = (n^2-1)(\alpha_n + \gamma_n) \tag{2.13}$$

subtracting (2.11) from (2.13) and adding (2.11) to (2.13), we get respectively

$$\alpha_{n+1} - \beta_{n+1} + \gamma_{n+1} = \frac{n+1}{n-1}\big(\alpha_n - \beta_n + \gamma_n\big),$$

$$\alpha_{n+1} + \beta_{n+1} + \gamma_{n+1} = \frac{n-1}{n+1}\big(\alpha_n + \beta_n + \gamma_n\big),$$

which implies

$$\alpha_n - \beta_n + \gamma_n = \left(\prod_{i=2}^{n-1} \frac{i+1}{i-1}\right) k_2 = \frac{n(n-1)}{2} k_2, \qquad k_2 \text{ is a constant,}$$

$$\alpha_n + \beta_n + \gamma_n = \left(\prod_{i=2}^{n-1} \frac{i-1}{i+1}\right) k_3 = \frac{2}{n(n-1)} k_3, \qquad k_3 \text{ is a constant.}$$

We have a linear system of difference equations for the coefficients α_n, β_n and γ_n,

$$\begin{aligned} \alpha_n \qquad - \gamma_n &= k_1, \\ \alpha_n - \beta_n + \gamma_n &= \frac{n(n-1)}{2} k_2, \\ \alpha_n + \beta_n + \gamma_n &= \frac{2}{n(n-1)} k_3, \end{aligned}$$

solving the system for the coefficients α_n, β_n and γ_n, hence

$$\begin{aligned} \alpha_n &= \frac{1}{2} k_1 + \frac{n(n-1)}{8} k_2 + \frac{1}{2n(n-1)} k_3, \\ \beta_n &= -\frac{n(n-1)}{4} k_2 + \frac{1}{n(n-1)} k_3, \\ \gamma_n &= -\frac{1}{2} k_1 + \frac{n(n-1)}{8} k_2 + \frac{1}{2n(n-1)} k_3, \end{aligned}$$

so the characteristic

$$\begin{aligned}
Q(n,u_n) &= \alpha_n u_n^2+\beta_n u_n+\gamma_n\\
&= k_1\big(\frac{1}{2}u_n^2-\frac{1}{2}\big)+k_2\Big(\frac{n(n-1)}{8}u_n^2-\frac{n(n-1)}{4}u_n+\frac{n(n-1)}{8}\Big)\\
&\qquad +k_3\Big(\frac{1}{2n(n-1)}u_n^2+\frac{1}{n(n-1)}u_n+\frac{1}{2n(n-1)}\Big)\\
&= \frac{1}{2}k_1(u_n^2-1)+\frac{n(n-1)}{8}k_2(u_n^2-2u_n+1)+\frac{1}{2n(n-1)}k_3(u_n^2+2u_n+1)
\end{aligned}$$

2.3.2 Second Order Difference Equations

For a second order difference equation,

$$u_{n+2}=\omega(n,u_n,u_{n+1}),$$

with a one-parameter local Lie group of symmetries, the symmetry condition is

$$\hat{u}_{n+2}=\omega(\hat{n},\hat{u}_{n+1},\hat{u}_n)\qquad when\qquad u_{n+2}=\omega(n,u_{n+1},u_n), \tag{2.14}$$

such that $\frac{\partial\omega}{\partial u_{n+1}}\neq 0$.

Substitute the Lie point symmetries of the form

$$\hat{n}=n,\qquad \hat{u}_{n+k}=u_{n+k}+\epsilon Q(n+k,u_{n+k})+O(\epsilon^2),\qquad \epsilon\in\mathbb{R} \tag{2.15}$$

into (2.14), we obtain

$$\begin{aligned}
\omega(\hat{n},\hat{u}_{n+1},\hat{u}_n) &= w\Big(n,u_{n+1}+\epsilon Q(n+1,u_{n+1}),u_n+\epsilon Q(n,u_n)\Big)\\
&= \omega(n,u_{n+1},u_n)+\epsilon\Big(\frac{\partial\omega}{\partial\hat{u}_{n+1}}\frac{\partial\hat{u}_{n+1}}{\partial\epsilon}\Big|_{\epsilon=0}+\frac{\partial\omega}{\partial\hat{u}_n}\frac{\partial\hat{u}_n}{\partial\epsilon}\Big|_{\epsilon=0}\Big)\\
&\qquad +O(\epsilon^2),\qquad \text{using a Taylor series about } \epsilon=0\\
&= \omega(n,u_{n+1},u_n)+\epsilon\Big(\frac{\partial\omega}{\partial u_{n+1}}Q(n+1,u_{n+1})+\frac{\partial\omega}{\partial u_n}Q(n,u_n)\Big)\\
&\qquad +O(\epsilon^2),
\end{aligned} \tag{2.16}$$

from (2.15) we have

$$\omega(\hat{n},\hat{u}_{n+1},\hat{u}_n)=\hat{u}_{n+2}=u_{n+2}+\epsilon Q(n+2,u_{n+2})+O(\epsilon^2), \tag{2.17}$$

comparing equations (2.16) and (2.17),we get the LSC for second order difference equation:

$$Q(n+2,u_{n+2}) = Q(n+2,\omega) = \frac{\partial\omega}{\partial u_{n+1}}Q(n+1,u_{n+1}) + \frac{\partial\omega}{\partial u_n}Q(n,u_n) \tag{2.18}$$

Definition 6. *[8]* The forward shift operator is defined by

$$S : n \mapsto n+1, \qquad S^i u_n = u_{n+i}.$$

Definition 7. *[8]* The infinitesimal generator X is

$$X = \sum_{i=0}^{k-1} \big(S^i Q(n,u_n)\big)\frac{\partial}{\partial u_{n+i}}.$$

The Linearized symmetry condition (2.18) can be written as

$$S^2 Q(n,u_n) = X\omega.$$

This LSC is a functional equation and it is hard to solve. Lie symmetries are diffeomorphism. Consequently, Q is a smooth function of its continuous arguments and so the LSC can be solved by the method of *differential elimination* as follow:

- First step: Eliminating the first term of LSC of the second order difference equation, $Q(n+2,u_{n+2})$, by applying the differential opartor (L),

$$\begin{aligned} L &= \frac{\partial}{\partial u_n} + \frac{\partial u_{n+1}}{\partial u_n}\frac{\partial}{\partial u_{n+1}} \\ &= \frac{\partial}{\partial u_n} - \frac{\partial\omega/\partial u_n}{\partial\omega/\partial u_{n+1}}\frac{\partial}{\partial u_{n+1}}, \end{aligned}$$

  to the LSC (2.18), since $\frac{\partial\omega}{\partial u_n} \neq 0$ and

$$\Big(\frac{\partial u_n}{\partial\omega}\frac{\partial}{\partial u_n} - \frac{\partial u_{n+1}}{\partial\omega}\frac{\partial}{\partial u_{n+1}}\Big)\big(Q(n+2,\omega)\big) = 0,$$

  then

$$\big(\frac{\partial}{\partial u_n} - \frac{\partial\omega/\partial u_n}{\partial\omega/\partial u_{n+1}}\frac{\partial}{\partial u_{n+1}}\Big)\big(Q(n+2,\omega)\big) = 0,$$

so

$$\Big(\frac{\partial}{\partial u_n}-\frac{\partial\omega/\partial u_n}{\partial\omega/\partial u_{n+1}}\frac{\partial}{\partial u_{n+1}}\Big)\big(\frac{\partial\omega}{\partial u_{n+1}}Q(n+1,u_{n+1})+\frac{\partial\omega}{\partial u_n}Q(n,u_n)\big)=0,$$

which implies

$$\begin{aligned}\frac{\partial^2\omega}{\partial u_{n+1}\partial u_n}Q(n+1,u_{n+1})-\frac{\partial\omega/\partial u_n}{\partial\omega/\partial u_{n+1}}\Big(\frac{\partial^2\omega}{\partial u_{n+1}^2}Q(n+1,u_{n+1})+\frac{\partial\omega}{\partial u_{n+1}}Q'(n+1,u_{n+1})\Big)\\ \frac{\partial^2\omega}{\partial u_n^2}Q(n,u_n)-\frac{\partial\omega/\partial u_n}{\partial\omega/\partial u_{n+1}}\Big(\frac{\partial^2\omega}{\partial u_{n+1}\partial u_n}Q(n,u_n)+\frac{\partial\omega}{\partial u_n}Q'(n,u_n)\Big) &= 0,\end{aligned}$$

the last functional equation dose not include $Q(n+2,u_{n+2})$.

- Second step: Elimination $Q(n+1,u_{n+1})$ and $Q'(n+1,u_{n+1})$. After appropriate calculations, differentiate once or more, as necessary, with respect to u_n keeping u_{n+1} fixed, then we get an ordinary differential equation, we solve it if possible and we obtain $Q(n,u_n)$ with undetermined coefficients as functions of n.

- Third step: To find these coefficients of the terms of $Q(n,u_n)$, we substitute $Q(n,u_n)$ in the equations that we obtained in previous steps which can be split into a system of linear difference equations by collecting all terms with the same dependence u_n and u_{n+1}.

Example 8. *[10]* Find the characteristics of the Lie point symmetries for the second order difference equation

$$u_{n+2}=e^{-n}u_{n+1}u_n^2=\omega(n,u_{n+1},u_n),$$

Solution. Differentiate ω with respect to u_{n+1},u_n

$$\begin{aligned}\frac{\partial\omega}{\partial u_{n+1}}&=e^{-n}u_n^2=\frac{\omega}{u_{n+1}},\\ \frac{\partial\omega}{\partial u_n}&=2e^{-n}u_{n+1}u_n=\frac{2\omega}{u_n}.\end{aligned}$$

then

$$\frac{\partial u_{n+1}}{\partial u_n}=-\frac{\partial\omega/\partial u_n}{\partial\omega/\partial u_{n+1}}=-\frac{2\omega/u_n}{\omega/u_{n+1}}=-\frac{2u_{n+1}}{u_n}$$

The LSC is

$$Q(n+2,u_{n+2})=\frac{\omega}{u_{n+1}}Q(n+1,u_{n+1})+\frac{2\omega}{u_n}Q(n,u_n),\tag{2.19}$$

which is functional equation. By differential elimination we transform it to differential equation as follows: apply the differential operator(L) on LSC functional equation

$$\begin{aligned} L &= \frac{\partial}{\partial u_n} + \frac{\partial u_{n+1}}{\partial u_n}\frac{\partial}{\partial u_{n+1}} \\ &= \frac{\partial}{\partial u_n} + \Big(-\frac{2u_{n+1}}{u_n}\Big)\frac{\partial}{\partial u_{n+1}}, \end{aligned}$$

$$\begin{aligned} 0 &= \frac{\partial}{\partial u_n}\Big(Q(n+2,u_{n+2})\Big) - \big(\frac{2u_{n+1}}{u_n}\big)\frac{\partial}{\partial u_{n+1}}\Big(Q(n+2,u_{n+2})\Big) \\ &= \frac{\partial}{\partial u_n}\Big(\frac{\omega}{u_{n+1}}Q(n+1,u_{n+1}) + \frac{2\omega}{u_n}Q(n,u_n)\Big) \\ &\qquad - \big(\frac{2u_{n+1}}{u_n}\big)\frac{\partial}{\partial u_{n+1}}\Big(\frac{\omega}{u_{n+1}}Q(n+1,u_{n+1}) + \frac{2\omega}{u_n}Q(n,u_n)\Big) \\ &= \frac{2\omega}{u_n}Q'(n,u_n) - \frac{2\omega}{u_n^2}Q(n,u_n) - \frac{2\omega}{u_n}Q'(n+1,u_{n+1}) + \frac{2\omega}{u_{n+1}u_n}Q(n+1,u_{n+1}) \end{aligned}$$

we get

$$\begin{aligned} 0 &= \frac{2\omega}{u_n}Q'(n,u_n) - \frac{2\omega}{u_n^2}Q(n,u_n) - \frac{2\omega}{u_n}Q'(n+1,u_{n+1}) + \frac{2\omega}{u_{n+1}u_n}Q(n+1,u_{n+1}) \\ &= Q'(n,u_n) - \frac{Q(n,u_n)}{u_n} - Q'(n+1,u_{n+1}) + \frac{Q(n+1,u_{n+1})}{u_{n+1}} \end{aligned} \tag{2.20}$$

Now, differentiate this equation with respect to u_n keeping u_{n+1} fixed,

$$\begin{aligned} 0 &= Q''(n,u_n) - \frac{Q'(n,u_n)}{u_n} + \frac{Q(n,u_n)}{u_n^2} \\ &= u_n^2 Q''(n,u_n) - u_n Q'(n,u_n) + Q(n,u_n) \end{aligned}$$

which is an Euler differential equation, whose solution is given by

$$Q(n,u_n) = \alpha_n u_n + \beta_n u_n \ln u_n,$$

where α and β are functions of n. Substitute this equation into (2.20),

$$\alpha_n + \beta_n + \beta_n \ln u_n - \alpha_n - \beta_n \ln u_n - \alpha_{n+1} - \beta_{n+1} - \beta_{n+1}\ln u_{n+1} + \alpha_{n+1} + \beta_{n+1}\ln u_{n+1} = 0,$$

then

$$\beta_n - \beta_{n+1} = 0,$$

which implies

$$\beta_n = c,$$

assume $c = \beta_n = 0$, then

$$Q(n, u_n) = \alpha_n u_n,$$

substitute this equation into LSC (2.19) in order to find α_n, and since $\omega = u_{n+2}$, we get

$$\alpha_{n+2} u_{n+2} = \frac{u_{n+2}}{u_{n+1}} \alpha_{n+1} u_{n+1} + \frac{2u_{n+2}}{u_n} \alpha_n u_n$$

which implies

$$\alpha_{n+2} - \alpha_{n+1} - 2\alpha_n = 0,$$

the characteristic equation is

$$r^2 - r - 2 = 0$$

and the characteristic roots are: $r = -1$ and $r = 2$, hence

$$\alpha_n = c_1(-1)^n + c_2(2)^n.$$

Thus,

$$Q(n, u_n) = \big(c_1(-1)^n + c_2(2)^n\big) u_n$$

2.4 Lie Symmetries to Solve Difference Equations

Finding $Q(n, u_n)$ in terms of u_n enables us to write the canonical coordinate in term of u_n if we admit the tanslation on canonical coordinate as follow:

$$\hat{n} = n, \qquad \hat{s}_n = s_n + \epsilon,$$

then the characteristic with respect to (n, s_n) is

$$\tilde{Q}(n, s_n) = 1 = s'(n, u_n) Q(n, u_n)$$

and so

$$s_n = \int \frac{1}{Q(n, u_n)}.$$

One of the main uses of a canonical coordinate is to simplify or even solve a given OΔE. The idea is to rewrite the OΔE as a simpler OΔE for s_n; if the simpler OΔE can be solved, all that remains is to write the solution in terms of the original variables. To use this approach, therefore, one must be able to invert the map from u_n to s_n. Any canonical coordinate s_n that meets this requirement will be called compatible with the OΔE.

2.4.1 First Order Difference Equations

It is not essential to use the general solution of $Q(n, u_n)$, a single solution can be used to find s_n and determine the solution of the difference equation. We suppose some constants for the general solution of $Q(n, u_n)$ equal zero. The following example, after Lemma, illustrate how the characteristic of the first order difference equation (2.9) can be used to find the solution.

LEMMA 2.4.1. Let u_{n_0} be given, then the general solution of the difference equation

$$u_{n+1} - u_n = f(n),$$

is given by

$$u_n = u_{n_0} + \sum_{k=n_0}^{n-1} f(k), \qquad \text{for } n > n_0.$$

Example 9. *[10]* Use lie point symmetry to solve the difference equation

$$u_{n+1} = \frac{nu_n + 1}{u_n + n}, \qquad n \geq 2, \qquad u_2 \geq -1$$

.

Solution. From example (7) and for $k_1 = 2,\ k_2 = 0$ and $k_3 = 0$,

$$Q(n, u_n) = u_n^2 - 1$$

there is no canonical coordinate $u_n = \pm 1$, if $u_2 = \pm 1$ then $u_n = u_2$. The appropriate real-valued canonical coordinate is

$$s_n = \int \frac{du_n}{u_n^2 - 1} = \begin{cases} \frac{1}{2} \ln \frac{u_n - 1}{u_n + 1}, & |u_n| > 1; \\ \frac{1}{2} \ln \frac{1 - u_n}{1 + u_n}, & |u_n| < 1, \end{cases}$$

but $u_2 \geq -1$ which implies $u_2 \in (-1, 1)$ or $(1, \infty)$ then u_n belong to the same interval, hence

$$s_n = \begin{cases} \frac{1}{2} \ln \frac{u_n - 1}{u_n + 1}, & u_n > 1; \\ \frac{1}{2} \ln \frac{1 - u_n}{1 + u_n}, & |u_n| < 1. \end{cases}$$

The transformation from u_n to s_n is not injective since $s_n(u_n) = s_n(\frac{1}{u_n})$, so s_n is not compatible canonical coordinate. To solve the difference equation and get u_n we seek an injective transformation to ensure the compatible condition.Therefore the problem of solving the difference equation splits into two separate parts.

Case 1: if $u_n > 1$, so

$$s_n = \frac{1}{2}\ln\frac{u_n - 1}{u_n + 1},$$

therefore the map from u_n to s_n is injective so the compatibility condition is satisfied and s_n is a compatible coordinate.

Now, consider the difference equation for s_n

$$\begin{aligned} s_{n+1} - s_n &= \frac{1}{2}\ln\Big(\frac{u_{n+1}-1}{u_{n+1}+1}\Big) - \frac{1}{2}\ln\Big(\frac{u_n-1}{u_n+1}\Big) \\ &= \frac{1}{2}\Big(\ln(u_{n+1}-1) - \ln(u_{n+1}+1) - \ln(u_n-1) + \ln(u_n+1)\Big) \\ &= \frac{1}{2}\Big(\ln\Big(\frac{nu_n+1}{u_n+n} - 1\Big) - \ln\Big(\frac{nu_n+1}{u_n+n} + 1\Big) - \ln(u_n-1) + \ln(u_n+1)\Big) \\ &= \frac{1}{2}\Big(\ln\Big(\frac{(u_n-1)(n-1)}{u_n+n}\Big) - \ln\Big(\frac{(u_n+1)(n+1)}{u_n+n}\Big) - \ln(u_n-1) + \ln(u_n+1)\Big) \\ &= \frac{1}{2}\ln\Big(\frac{n-1}{n+1}\Big), \end{aligned}$$

then

$$\begin{aligned} s_n &= s_2 + \frac{1}{2}\sum_{k=2}^{n-1}\ln\Big(\frac{k-1}{k+1}\Big) \\ &= \frac{1}{2}\ln\Big(\frac{u_2-1}{u_2+1}\Big) + \frac{1}{2}\ln\Big(\prod_{k=2}^{n-1}\frac{k-1}{k+1}\Big) \\ &= \frac{1}{2}\ln\Big(\frac{u_2-1}{u_2+1}\Big) + \frac{1}{2}\ln\Big(\frac{2}{n(n-1)}\Big) \\ &= \frac{1}{2}\ln\Big(\frac{2(u_2-1)}{(u_2+1)n(n-1)}\Big), \end{aligned}$$

so

$$\frac{1}{2}\ln\Big(\frac{u_n-1}{u_n+1}\Big) = \frac{1}{2}\ln\Big(\frac{2(u_2-1)}{(u_2+1)n(n-1)}\Big)$$

which implies

$$u_n = \frac{(u_2+1)n(n-1) + 2(u_2-1)}{(u_2+1)n(n-1) - 2(u_2-1)}.$$

case 2: if $|u_n| < 1$, so

$$s_n = \frac{1}{2}\ln\left(\frac{1-u_n}{1+u_n}\right),$$

therefore the map from u_n to s_n is injective so s_n is a compatible coordinate.

$$\begin{aligned} s_{n+1} - s_n &= \frac{1}{2}\ln\left(\frac{1-u_{n+1}}{1+u_{n+1}}\right) - \frac{1}{2}\ln\left(\frac{1-u_n}{1+u_n}\right) \\ &= \frac{1}{2}\ln\left(\frac{n-1}{n+1}\right), \end{aligned}$$

then

$$\begin{aligned} s_n &= s_2 + \frac{1}{2}\sum_{k=2}^{n-1}\ln\left(\frac{k-1}{k+1}\right) \\ &= \frac{1}{2}\ln\left(\frac{2(1-u_2)}{(1+u_2)n(n-1)}\right), \end{aligned}$$

so

$$\frac{1}{2}\ln\left(\frac{1-u_n}{1+u_n}\right) = \frac{1}{2}\ln\left(\frac{2(1-u_2)}{(1+u_2)n(n-1)}\right)$$

which implies

$$u_n = \frac{(u_2+1)n(n-1)+2(u_2-1)}{(u_2+1)n(n-1)-2(u_2-1)}.$$

thus, this value of u_n is valid for all $u_n \geq -1$. The general solution happens to include the solutions on which $Q(n, u_n) = 0$.

2.4.2 Second Order Difference Equations

Using Lie point symmetries (Characteristics) to solve a second order difference equation is similar to Lie point symmetries for solving first order difference equations. In addition we need to utilize the invariant function which can reduce the order of the difference equation by *one*.

Definition 8. *[8]* A function v_n is invariant under the Lie group of transformations Γ_ϵ if $X(v_n) = 0$, where X is the infinitesimal generator mentioned in definition (7).

Consider the characteristic $Q(n, u_n)$ for the second order difference equation

$$u_{n+2} = \omega(n, u_n, u_{n+1})$$

is known, the invariant v_n can be found by solving the quasi linear partial differential equation

$$Xv_n = Q(n, u_n)\frac{\partial v_n}{\partial u_n} + Q(n, u_{n+1})\frac{\partial v_n}{\partial u_{n+1}} = 0,$$

then

$$\frac{du_n}{Q(n, u_n)} = \frac{du_{n+1}}{Q(n, u_{n+1})} = \frac{dv_n}{0},$$

that can be solved using the characteristic method. If the invariant function $v_{n+1}(n, u_n, u_{n+1})$ can be written as a function of n and v_n only, then v_n can reduce the order of the difference equation by one, and we get $u_{n+1} = f(n, u_n, v_n)$. The following example shows how we can reduce the second order difference equation by one and solve a difference equation for s_n to obtain the solution of the second order difference equation.

Example 10. Consider the second order difference equation

$$u_{n+2} = e^{-n}u_{n+1}u_n^2,$$

using the characteristic obtained from example (8) to determine the solution of this difference equation.

Solution. from example (8), suppose $c_1 = 1$ and $c_2 = 0$, we get

$$Q(n, u_n) = (-1)^n u_n$$

Now we want to find the invariant using,

$$\frac{du_n}{(-1)^n u_n} = \frac{du_{n+1}}{(-1)^{n+1}u_{n+1}} := \frac{dv_n}{0},$$

take $\frac{du_n}{(-1)^n u_n} = \frac{du_{n+1}}{(-1)^{n+1}u_{n+1}}$, then

$$\ln \mid u_n \mid = -\ln \mid u_{n+1} \mid +c, \quad \text{then} \quad c = \ln \mid u_n u_{n+1} \mid$$

where c is a constant, so

$$k_1 = u_n u_{n+1} \text{ where } k_1 = e^c,$$

we also have

$$\frac{du_n}{u_n} := \frac{dv_n}{0}$$

then

$$v_n = k, \quad \text{such that} \quad k = f(k_1),$$

where k, k_1 are constants. Let $f(k_1) = k_1$, then

$$v_n = u_{n+1}u_n,$$

and

$$\begin{aligned} v_{n+1} &= u_{n+2}u_{n+1} \\ &= e^{-n}u_{n+1}u_n^2u_{n+1} \\ &= e^{-n}v_n^2. \end{aligned}$$

Now, we want to solve this equation recursively, let v_0 be given then

$$\begin{aligned}
&v_1 = v_0^2 \\
&v_2 = e^{-1}v_1^2 && = e^{-1}v_0^4 \\
&v_3 = e^{-2}v_2^2 = e^{-2}(e^{-1}v_0^4)^2 && = e^{-4}v_0^8 \\
&v_4 = e^{-3}v_3^2 = e^{-3}(e^{-2}e^{-1\cdot 2}v_0^8)^2 && = e^{-11}v_0^{16} \\
&v_5 = e^{-4}v_4^2 = e^{-4}(e^{-3}e^{-2\cdot 2}e^{-1\cdot 4}v_0^{16})^2 && = e^{-24}v_0^{32} \\
&v_6 = e^{-5}v_5^2 = e^{-5}(e^{-4}e^{-3\cdot 2}e^{-2\cdot 4}e^{-1\cdot 8}v_0^{32})^2 && = e^{-57}v_0^{64} \\
&\quad\vdots
\end{aligned}$$

hence,

$$v_n = e^{\sum_{i=0}^{n-2}(1-n+i)2^i}v_0^{2^n} = u_{n+1}u_n$$

thus,

$$u_{n+1} = \frac{1}{u_n}e^{\sum_{i=0}^{n-2}(1-n+i)2^i}(u_1u_0)^{2^n}.$$

The order of the difference equation has been reduced by one. To solve the last equation we need to obtain the canonical coordinate,

$$s_n = \int \frac{du_n}{(-1)^n u_n} = (-1)^n \ln|u_n|,$$

then

$$\begin{aligned} s_{n+1}-s_n &= (-1)^{n+1}\ln|u_{n+1}|-(-1)^n\ln|u_n| \\ &= (-1)^{n+1}[\ln|u_{n+1}|+\ln|u_n|] \\ &= (-1)^{n+1}\ln|u_{n+1}u_n| \\ &= (-1)^{n+1}\ln|v_n| \\ &= (-1)^{n+1}\ln\left|e^{\sum_{i=0}^{n-2}(1-n+i)2^i}(u_1u_0)^{2^n}\right| \end{aligned}$$

so

$$s_{n+1}-s_n=(-1)^{n+1}\ln\left(e^{\sum_{i=0}^{n-2}(1-n+i)2^i}(u_1u_0)^{2^n}\right)$$

which is a first order homogeneous difference equation. Let s_0 be given and by lemma (2.4.1), we get

$$\begin{aligned} s_n &= s_0+\sum_{k=0}^{n-1}(-1)^{k+1}\ln\left(e^{\sum_{i=0}^{k-2}(1-k+i)2^i}(u_1u_0)^{2^k}\right) \\ &= \ln|u_0|+\sum_{k=0}^{n-1}\ln\left(e^{\sum_{i=0}^{k-2}(1-k+i)2^i}(u_1u_0)^{2^k}\right)^{(-1)^{k+1}} \\ &= \ln\left(|u_0|\prod_{k=0}^{n-1}\left(e^{\sum_{i=0}^{k-2}(1-k+i)2^i}(u_1u_0)^{2^k}\right)^{(-1)^{k+1}}\right) \end{aligned}$$

The canonical coordinate is

$$s_n=(-1)^n\ln|u_n|$$

which implies

$$\begin{aligned} u_n &= \exp\left[(-1)^n s_n\right] \\ &= \exp\left[(-1)^n\ln\left(|u_0|\prod_{k=0}^{n-1}\left(e^{\sum_{i=0}^{k-2}(1-k+i)2^i}(u_1u_0)^{2^k}\right)^{(-1)^{k+1}}\right)\right] \\ &= \exp\left[\ln\left(|u_0|^{(-1)^n}\prod_{k=0}^{n-1}\left(e^{\sum_{i=0}^{k-2}(1-k+i)2^i}(u_1u_0)^{2^k}\right)^{(-1)^{k+n+1}}\right)\right] \\ &= |u_0|^{(-1)^n}\prod_{k=0}^{n-1}\left(e^{\sum_{i=0}^{k-2}(1-k+i)2^i}(u_1u_0)^{2^k}\right)^{(-1)^{k+n+1}} \end{aligned}$$

Example 11. Solve the following O∆E by finding the characteristic of Lie point symmetry and using it for reduction of order:

$$u_{n+2} = \frac{u_n}{a + bu_{n+1}u_n} = w \tag{2.21}$$

Solution. Let us differentiate this difference equation with respect to u_n and u_{n+1}

$$\begin{aligned}
\frac{\partial w}{\partial u_n} &= \frac{a + bu_n u_{n+1} - bu_n u_{n+1}}{(a + bu_n u_{n+3})^2} = \frac{a}{(a + bu_n u_{n+1})^2} . \frac{u_n^2}{u_n^2} = \frac{aw^2}{u_n^2}, \\
\frac{\partial w}{\partial u_{n+1}} &= \frac{-bu_n^2}{(a + bu_n u_{n+1})^2} = -bw^2
\end{aligned}$$

and so

$$\frac{\partial u_{n+1}}{\partial u_n} = -\frac{\partial w/\partial u_n}{\partial w/\partial u_{n+1}} = -\frac{aw^2/u_n^2}{-bw^2} = \frac{a}{bu_n^2}.$$

The linearized symmetry condition (LSC) is given by

$$Q(n+2, u_{n+2}) - \frac{\partial w}{\partial u_{n+1}} Q(n+1, u_{n+1}) - \frac{\partial w}{\partial u_n} Q(n, u_n) = 0$$

$$Q(n+2, u_{n+2}) + bw^2 Q(n+1, u_{n+1}) - \frac{aw^2}{u_n^2} Q(n, u_n) = 0.$$

Now, by applying the differential operator L to the previous equation, where

$$L = \frac{\partial}{\partial u_n} + \frac{\partial u_{n+1}}{\partial u_n} \frac{\partial}{\partial u_{n+1}}$$

we get

$$\begin{aligned}
&\frac{\partial}{\partial u_n}\Big(Q(n+2, u_{n+2})\Big) + \frac{\partial u_{n+1}}{\partial u_n} \frac{\partial}{\partial u_{n+1}}\Big(Q(n+2, u_{n+2})\Big) \\
&= \frac{\partial}{\partial u_n}\Big(- bw^2 Q(n+1, u_{n+1}) + \frac{aw^2}{u_n^2} Q(n, u_n)\Big) \\
&\qquad\qquad + \frac{a}{bu_n^2} \frac{\partial}{\partial u_{n+1}}\Big(- bw^2 Q(n+1, u_{n+1}) + \frac{aw^2}{u_n^2} Q(n, u_n)\Big) \\
&= \frac{aw^2}{u_n^2} Q'(n, u_n) - \frac{2aw^2}{u_n^3} Q(n, u_n) - \frac{aw^2}{u_n^2} Q'(n+1, u_{n+1}) \\
&= 0
\end{aligned}$$

multiply this equation by $\frac{u_n^2}{aw^2}$

$$Q'(n,u_n) - \frac{2}{u_n}Q(n,u_n) - Q'(n+1,u_{n+1}) = 0 \tag{2.22}$$

differentiate the equation with respect to u_n keeping u_{n+1} fixed .

$$Q''(n,u_n) - \frac{2}{u_n}Q'(n,u_n) + \frac{2}{u_n^2}Q(n,u_n) = 0$$

again multiply by u_n^2

$$u_n^2 Q''(n,u_n) - 2u_n Q'(n,u_n) + 2Q(n,u_n) = 0$$

which is an Euler Equation, whose solution is

$$Q(n,u_n) = \alpha(n)u_n^2 + \beta(n)u_n$$

thus,

$$Q'(n,u_n) = 2\alpha(n)u_n + \beta(n)$$

substitute into equation (2.22)

$$\begin{aligned} 0 &= 2\alpha(n)u_n + \beta(n) - 2\alpha(n)u_n - 2\beta(n) - 2\alpha(n+1)u_{n+1} - \beta(n+1) \\ &= -\beta(n) - \beta(n+1) - 2\alpha(n+1)u_{n+1} \end{aligned}$$

comparing both sides of the last equation, we have

$$\alpha(n+1) = 0 \text{ and so } \alpha(n) = 0$$

and we have also

$$\beta(n+1) + \beta(n) = 0$$

which is a first order linear difference equation, whose solution is

$$\beta(n) = c(-1)^n$$

where c is constant. Suppose that $c = 1$ so $\beta(n) = (-1)^n$, which implies

$$Q(n,u_n) = (-1)^n u_n.$$

We want to find the invariant using,

$$\frac{du_n}{(-1)^n u_n} = \frac{du_{n+1}}{(-1)^{n+1} u_{n+1}} := \frac{dv_n}{0},$$

take $\frac{du_n}{(-1)^n u_n} = \frac{du_{n+1}}{(-1)^{n+1} u_{n+1}}$ invariants,
so

$$\ln \mid u_n \mid = -\ln \mid u_{n+1} \mid + c, \quad \text{then} \quad c = \ln \mid u_n u_{n+1} \mid$$

where c is constant, so

$$k_1 = u_n u_{n+1} \text{ where } k_1 = e^c,$$

we also have

$$\frac{du_n}{u_n} := \frac{dv_n}{0}$$

and so

$$v_n = k, \quad \text{such that} \quad k = f(k_1), \qquad \text{f is an arbitrary function}$$

where k, k_1 are constants. Let $f(k_1) = k_1$, then

$$v_n = u_{n+1} u_n,$$

and

$$v_{n+1} = u_{n+2} u_{n+1} = \frac{u_n u_{n+1}}{a + b u_n u_{n+1}} = \frac{v_n}{a + b v_n}.$$

Now, we want to solve this equation

$$v_{n+1} = \frac{v_n}{a + b v_n} \tag{2.23}$$

so

$$\frac{1}{v_{n+1}} = \frac{a}{v_n} + b,$$

let

$$z_n = \frac{1}{v_n},$$

this substitution converts equation (2.23) to the following first order linear equation

$$z_{n+1} - a z_n - b = 0$$

whose solution is

$$z_n = \left\{ \begin{array}{ll} a^n z_0 + b\left[\frac{a^n-1}{a-1}\right], & \text{if } a \neq 1 \\ z_0 + bn, & \text{if } a = 1 \end{array} \right\}$$

Case 1 : if $a \neq 1$

$$z_n = a^n z_0 + b\left[\frac{a^n - 1}{a - 1}\right]$$

so

$$v_n = \frac{1}{z_n} = \frac{1}{a^n z_0 + b\left[\frac{a^n-1}{a-1}\right]} = u_n u_{n+1},$$

thus

$$u_{n+1} = \frac{1}{u_n(a^n z_0 + b\left[\frac{a^n-1}{a-1}\right])}.$$

To solve the last equation we need to obtain the canonical coordinate,

$$\begin{aligned} s_n &= \int \frac{du_n}{(-1)^n u_n} \\ &= (-1)^n \ln |u_n| \,. \end{aligned}$$

So

$$\begin{array}{rcl} s_{n+1} - s_n & = & (-1)^{n+1} \ln |u_{n+1}| - (-1)^n \ln |u_n| \\ & = & (-1)^{n+1} [\ln |u_{n+1}| + \ln |u_n|] \\ & = & (-1)^{n+1} \ln |u_{n+1} u_n| \\ & = & (-1)^{n+1} \ln |v_n| \\ & = & (-1)^{n+1} \ln \left| \dfrac{1}{a^n z_0 + b\left[\frac{a^n-1}{a-1}\right]} \right| \\ & = & (-1)^n \ln \left| a^n z_0 + b(\dfrac{a^n - 1}{a - 1}) \right| \end{array}$$

so

$$s_{n+1} - s_n = (-1)^n \ln \left| a^n z_0 + b(\frac{a^n - 1}{a - 1}) \right|$$

, which is a first order nonhomogenous difference equation, by lemma 2.4.1

then

$$\begin{aligned} s_n &= s_0 + \sum_{k=0}^{n-1} (-1)^k \ln \left| a^k z_0 + b\left(\frac{a^k - 1}{a - 1}\right) \right| \\ &= \ln |u_0| + \sum_{k=0}^{n-1} (-1)^k \ln \left| a^k z_0 + b\left(\frac{a^k - 1}{a - 1}\right) \right| \end{aligned}$$

The canonical coordinate is

$$s_n = (-1)^n \ln |u_n|$$

which implies

$$\begin{aligned} u_n &= \exp\big((-1)^n s_n\big) \\ &= \exp\left((-1)^n \ln |u_0| + (-1)^n \sum_{k=0}^{n-1} (-1)^k \ln \left| a^k z_0 + b\left[\frac{a^k - 1}{a - 1}\right] \right| \right) \\ &= \exp\left(\ln |u_0|^{(-1)^n} \right) \exp\left(\sum_{k=0}^{n-1} (-1)^{n+k} \ln \left| a^k z_0 + b\left[\frac{a^k - 1}{a - 1}\right] \right| \right) \\ &= u_0^{(-1)^n} \quad \exp\left(\sum_{k=0}^{n-1} \ln \left(a^k z_0 + b\left[\frac{a^k - 1}{a - 1}\right] \right)^{(-1)^{k+n}} \right) \\ &= u_0^{(-1)^n} \quad \prod_{k=0}^{n-1} \left(a^k \frac{1}{u_0 u_1} + b\left[\frac{a^k - 1}{a - 1}\right] \right)^{(-1)^{k+n}} . \end{aligned} \tag{2.24}$$

To verify our computations, we want to show that solution (2.24) satisfy equation (2.21), which can be written as

$$\frac{u_n}{u_{n+2}} = a + b u_{n+1} u_n$$

The left hand side

$$\begin{aligned}
\frac{u_n}{u_{n+2}} &= \frac{u_0^{(-1)^n} \quad \prod_{k=0}^{n-1}\left(a^k \frac{1}{u_0u_1} + b\left[\frac{a^k-1}{a-1}\right]\right)^{(-1)^{k+n}}}{u_0^{(-1)^{n+2}} \quad \prod_{k=0}^{n+1}\left(a^k \frac{1}{u_0u_1} + b\left[\frac{a^k-1}{a-1}\right]\right)^{(-1)^{k+n+2}}}\\
&= \frac{u_0^{(-1)^n} \quad \prod_{k=0}^{n-1}\left(a^k \frac{1}{u_0u_1} + b\left[\frac{a^k-1}{a-1}\right]\right)^{(-1)^{k+n}}}{u_0^{(-1)^{n}} \quad \prod_{k=0}^{n+1}\left(a^k \frac{1}{u_0u_1} + b\left[\frac{a^k-1}{a-1}\right]\right)^{(-1)^{k+n}}}\\
&= \frac{\prod_{k=0}^{n-1}\left(a^k \frac{1}{u_0u_1} + b\left[\frac{a^k-1}{a-1}\right]\right)^{(-1)^{k+n}}}{\prod_{k=0}^{n-1}\left(a^k \frac{1}{u_0u_1} + b\left[\frac{a^k-1}{a-1}\right]\right)^{(-1)^{k+n}}}\\
&\qquad \cdot \frac{1}{\left(a^n \frac{1}{u_0u_1} + b\left[\frac{a^n-1}{a-1}\right]\right)^{(-1)^{n+n}}\left(a^{n+1} \frac{1}{u_0u_1} + b\left[\frac{a^{n+1}-1}{a-1}\right]\right)^{(-1)^{n+1+n}}}\\
&= \frac{1}{\left(a^n \frac{1}{u_0u_1} + b\left[\frac{a^n-1}{a-1}\right]\right)^{(-1)^{2n}}\left(a^{n+1} \frac{1}{u_0u_1} + b\left[\frac{a^{n+1}-1}{a-1}\right]\right)^{(-1)^{2n+1}}}\\
&= \frac{\left(a^{n+1} \frac{1}{u_0u_1} + b\left[\frac{a^{n+1}-1}{a-1}\right]\right)}{\left(a^n \frac{1}{u_0u_1} + b\left[\frac{a^n-1}{a-1}\right]\right)}.
\end{aligned}$$

The right hand side $a + bu_{n+1}u_n$

$$\begin{aligned}
u_{n+1}u_n &= u_0^{(-1)^{n+1}} \quad \prod_{k=0}^{n}\left(a^k \frac{1}{u_0u_1} + b\left[\frac{a^k-1}{a-1}\right]\right)^{(-1)^{k+n+1}} \quad u_0^{(-1)^n} \quad \prod_{k=0}^{n-1}\left(a^k \frac{1}{u_0u_1} + b\left[\frac{a^k-1}{a-1}\right]\right)^{(-1)^{k+n}}\\
&= u_0^{(-1)^{n+1}} \; u_0^{(-1)^n} \; \prod_{k=0}^{n}\left(a^k \frac{1}{u_0u_1} + b\left[\frac{a^k-1}{a-1}\right]\right)^{(-1)^{k+n+1}} \prod_{k=0}^{n-1}\left(a^k \frac{1}{u_0u_1} + b\left[\frac{a^k-1}{a-1}\right]\right)^{(-1)^{k+n}}\\
&= \prod_{k=0}^{n-1}\left(a^k \frac{1}{u_0u_1} + b\left[\frac{a^k-1}{a-1}\right]\right)^{(-1)^{k+n+1}} \left(a^n \frac{1}{u_0u_1} + b\left[\frac{a^n-1}{a-1}\right]\right)^{(-1)^{n+n+1}}\\
&\qquad \cdot \prod_{k=0}^{n-1}\left(a^k \frac{1}{u_0u_1} + b\left[\frac{a^k-1}{a-1}\right]\right)^{(-1)^{k+n}}\\
&= \left(a^n \frac{1}{u_0u_1} + b\left[\frac{a^n-1}{a-1}\right]\right)^{(-1)^{2n+1}}\\
&= \left(a^n \frac{1}{u_0u_1} + b\left[\frac{a^n-1}{a-1}\right]\right)^{-1}
\end{aligned}$$

so

$$\begin{aligned} a + bu_{n+1}u_n &= a + \frac{b}{\left(a^n \frac{1}{u_0u_1} + b\left[\frac{a^n-1}{a-1}\right]\right)} \\ &= \frac{a^{n+1}\frac{1}{u_0u_1} + \frac{ba^{n+1}-ba}{a-1} + b}{\left(a^n \frac{1}{u_0u_1} + b\left[\frac{a^n-1}{a-1}\right]\right)} \\ &= \frac{\left(a^{n+1}\frac{1}{u_0u_1} + b\left[\frac{a^{n+1}-1}{a-1}\right]\right)}{\left(a^n \frac{1}{u_0u_1} + b\left[\frac{a^n-1}{a-1}\right]\right)} \end{aligned}$$

Case 2 : if $a = 1$

$$z_n = z_0 + bn$$

so

$$v_n = \frac{1}{z_n} = \frac{1}{z_0 + bn} = u_n u_{n+1}$$

and so

$$u_{n+1} = \frac{1}{u_n(z_0 + bn)}.$$

To solve the last equation we need to obtain the canonical coordinate,

$$s_n = \int \frac{du_n}{(-1)^n u_n}$$

$$= (-1)^n \ln |u_n|$$

so

$$\begin{aligned} s_{n+1} - s_n &= (-1)^{n+1} \ln |u_{n+1}| - (-1)^n \ln |u_n| \\ &= (-1)^{n+1}[\ln |u_{n+1}| + \ln |u_n|] \\ &= (-1)^{n+1} \ln |u_{n+1}u_n| \\ &= (-1)^{n+1} \ln |v_n| \\ &= (-1)^{n+1} \ln \left|\frac{1}{z_0 + bn}\right| \\ &= (-1)^n \ln |z_0 + bn| \end{aligned}$$

so

$$s_{n+1} - s_n = (-1)^n \ln|z_0 + bn|,$$

which is a first order nonhomogenous difference equation, by lemma 2.4.1, then

$$\begin{aligned} s_n &= s_0 + \sum_{k=0}^{n-1} (-1)^k \ln|z_0 + bk| \\ &= \ln|u_0| + \sum_{k=0}^{n-1} (-1)^k \ln|z_0 + bk| \end{aligned}$$

The canonical coordinate is

$$s_n = (-1)^n \ln|u_n|$$

which implies

$$\begin{aligned} u_n &= \exp\big((-1)^n s_n\big) \\ &= \exp\left((-1)^n \ln|u_0| + (-1)^n \sum_{k=0}^{n-1} (-1)^k \ln|z_0 + bk|\right) \\ &= \exp\left(\ln|u_0|^{(-1)^n}\right) \exp\left(\sum_{k=0}^{n-1} (-1)^{n+k} \ln|z_0 + bk|\right) \\ &= u_0^{(-1)^n} \exp\left(\sum_{k=0}^{n-1} \ln\big(z_0 + bk\big)^{(-1)^{k+n}}\right) \\ &= u_0^{(-1)^n} \prod_{k=0}^{n-1} \big(\frac{1}{u_0 u_1} + bk\big)^{(-1)^{k+n}}. \end{aligned} \tag{2.25}$$

verifying the solution as in the previous case. As mentioned above, equation (2.21) can be written as

$$\frac{u_n}{u_{n+2}} = a + b u_{n+1} u_n.$$

The left hand side

$$
\begin{aligned}
\frac{u_n}{u_{n+2}} &= \frac{u_0^{(-1)^n} \cdot \prod_{k=0}^{n-1} \left(\frac{1}{u_0u_1} + bk\right)^{(-1)^{k+n}}}{u_0^{(-1)^{n+2}} \cdot \prod_{k=0}^{n+1} \left(\frac{1}{u_0u_1} + bk\right)^{(-1)^{k+n+2}}} \\
&= \frac{u_0^{(-1)^n} \cdot \prod_{k=0}^{n-1} \left(\frac{1}{u_0u_1} + bk\right)^{(-1)^{k+n}}}{u_0^{(-1)^n} \cdot \prod_{k=0}^{n+1} \left(\frac{1}{u_0u_1} + bk\right)^{(-1)^{k+n}}} \\
&= \frac{\prod_{k=0}^{n-1} \left(\frac{1}{u_0u_1} + bk\right)^{(-1)^{k+n}}}{\prod_{k=0}^{n+1} \left(\frac{1}{u_0u_1} + bk\right)^{(-1)^{k+n}}} \\
&= \frac{\prod_{k=0}^{n-1} \left(\frac{1}{u_0u_1} + bk\right)^{(-1)^{k+n}}}{\prod_{k=0}^{n-1} \left(\frac{1}{u_0u_1} + bk\right)^{(-1)^{k+n}} \cdot \left(\frac{1}{u_0u_1} + bn\right)^{(-1)^{n+n}} \cdot \left(\frac{1}{u_0u_1} + b(n+1)\right)^{(-1)^{n+1+n}}} \\
&= \frac{1}{\left(\frac{1}{u_0u_1} + bn\right)^{(-1)^{2n}} \cdot \left(\frac{1}{u_0u_1} + b(n+1)\right)^{(-1)^{2n+1}}} \\
&= \frac{\frac{1}{u_0u_1} + b(n+1)}{\frac{1}{u_0u_1} + bn}.
\end{aligned}
$$

Now, calculating $a + bu_{n+1}u_n$

$$
\begin{aligned}
u_{n+1}u_n &= u_0^{(-1)^{n+1}} \prod_{k=0}^{n} \left(\frac{1}{u_0u_1} + bk\right)^{(-1)^{k+n+1}} u_0^{(-1)^n} \prod_{k=0}^{n-1} \left(\frac{1}{u_0u_1} + bk\right)^{(-1)^{k+n}} \\
&= u_0^{(-1)^{n+1}} u_0^{(-1)^n} \prod_{k=0}^{n-1} \left(\frac{1}{u_0u_1} + bk\right)^{(-1)^{k+n+1}} \left(\frac{1}{u_0u_1} + bn\right)^{(-1)^{n+n+1}} \prod_{k=0}^{n-1} \left(\frac{1}{u_0u_1} + bk\right)^{(-1)^{k+n}} \\
&= \left(\frac{1}{u_0u_1} + bn\right)^{(-1)^{2n+1}} \\
&= \left(\frac{1}{u_0u_1} + bn\right)^{-1}
\end{aligned}
$$

so

$$\begin{aligned} a + bu_{n+1}u_n &= 1 + bu_{n+1}u_n = 1 + \frac{b}{\left(\frac{1}{u_0u_1} + bn\right)} \\ &= \frac{\frac{1}{u_0u_1} + bn + b}{\left(\frac{1}{u_0u_1} + bn\right)} \\ &= \frac{\frac{1}{u_0u_1} + b(n+1)}{\frac{1}{u_0u_1} + bn}. \end{aligned}$$

So our computations are true.

2.5 Higher Order Difference Equations

In order to solve an OΔE of order k,

$$u_{n+k} = \omega(n, u, ..., u_{n+k-1}), \qquad \frac{\partial \omega}{\partial u_n} \neq 0$$

using symmetry method, apply similar steps as those to solve second order difference equation in the previous sections:

- Step 1: Write out the LSC for the OΔE which is a functional equation.
- Step 2: Using differential elimination with appropriate differential operators and suitable differentiation with respect independent variable to solve the LSC to get a differential equation for $Q(n, u_n)$, and solve it if that is possible.
- Step 3: Substitute $Q(n, u_n)$ in equations obtained from step 2 and the LSC to get the coefficients of terms of $Q(n, u_n)$.
- Step 4: After finding the characteristic $Q(n, u_n)$, we want to get the invariant v_n by solving the partial differential equation

  $$Xv_n = Q(n, u_n)\frac{\partial v_n}{\partial u_n} + SQ(n, u_n)\frac{\partial v_n}{\partial u_{n+1}} + \cdots + S^{p-1}Q(n, u_n)\frac{\partial v_n}{\partial u_{n+p-1}},$$

  solving this equation using characteristic method, we set

  $$\frac{du_n}{Q(n, u_n)} = \frac{du_{n+1}}{SQ(n, u_n)} = \cdots = \frac{du_{n+p-1}}{S^{p-1}Q(n, u_n)} := \frac{dv_n}{0}$$

- Step 5: Write a compatible canonical coordinate s_n and solve a difference equation for s_n that would be written after finding the invariant
- Step 6: We get u_n from the canonical coordinate

3. FORBIDDEN SET OF THE DIFFERENCE EQUATION $U_{N+2(I+1)} = \frac{U_N}{A+BU_{N+I+1}U_N}$

In this chapter we will find a closed form solution for the difference equation

$$u_{n+2(i+1)} = \frac{u_n}{a + bu_{n+i+1}u_n}$$

for even i. Then, we give full details for a special case $i = 2$ and for $i = 0$. Then we solve (open problem 17 for even i in [3]) by finding the forbidden set of these difference equations. We assume that $(u_0, u_1, u_2, \ldots, u_{2r-1}) \in \mathbb{R}^{2r}$ such that $u_0u_1u_2\ldots u_{2r-1} \neq 0$ and $a, b > 0$

3.1 Solution of $u_{n+2(i+1)} = \frac{u_n}{a+bu_{n+i+1}u_n}$ *When* i *is Even*

Consider the difference equation

$$u_{n+2(i+1)} = \frac{u_n}{a + bu_{n+i+1}u_n} = w(u_n, u_{n+i+1}), \quad u_0u_1u_2\ldots u_{2r-1} \neq 0. \tag{3.1}$$

Let $r = i + 1$, be an odd number then

$$u_{n+2r} = \frac{u_n}{a + bu_{n+r}u_n} = w, \tag{3.2}$$

Differentiate w with respect to u_n and u_{n+r}

$$\begin{aligned}
\frac{\partial w}{\partial u_n} &= \frac{a + bu_nu_{n+r} - bu_nu_{n+r}}{(a + bu_nu_{n+r})^2} = \frac{a}{(a + bu_nu_{n+r})^2} \cdot \frac{u_n^2}{u_n^2} = \frac{aw^2}{u_n^2}, \\
\frac{\partial w}{\partial u_{n+r}} &= \frac{-bu_n^2}{(a + bu_nu_{n+r})^2} = -bw^2
\end{aligned}$$

and so

$$\frac{\partial u_{n+r}}{\partial u_n} = -\frac{\partial w/\partial u_n}{\partial w/\partial u_{n+r}} = -\frac{aw^2/u_n^2}{-bw^2} = \frac{a}{bu_n^2}.$$

The linearized symmetry condition (LSC) is given by

$$Q(n+2r, u_{n+2r}) - \frac{\partial w}{\partial u_{n+r}} Q(n+r, u_{n+r}) - \frac{\partial w}{\partial u_n} Q(n, u_n) = 0$$

$$Q(n+2r, u_{n+2r}) + bw^2 Q(n+r, u_{n+r}) - \frac{aw^2}{u_n^2} Q(n, u_n) = 0.$$

Now, by applying the differential operator L to the previous equation, where

$$\begin{aligned} L &= \frac{\partial}{\partial u_n} + \frac{\partial u_{n+r}}{\partial u_n} \frac{\partial}{\partial u_{n+r}} \\ &= \frac{\partial}{\partial u_n} + \frac{a}{bu_n^2} \frac{\partial}{\partial u_{n+r}} \end{aligned}$$

we get

$$\frac{\partial}{\partial u_n}\Big(Q(n+2r, u_{n+2r})\Big) + \frac{a}{bu_n^2}\frac{\partial}{\partial u_{n+r}}\Big(Q(n+2r, u_{n+2r})\Big)$$

$$\begin{aligned} = \ & \frac{\partial}{\partial u_n}\Big(-bw^2 Q(n+r, u_{n+r}) + \frac{aw^2}{u_n^2} Q(n, u_n)\Big) \\ & + \frac{a}{bu_n^2}\frac{\partial}{\partial u_{n+r}}\Big(-bw^2 Q(n+r, u_{n+r}) + \frac{aw^2}{u_n^2} Q(n, u_n)\Big) \end{aligned}$$

which implies

$$\frac{aw^2}{u_n^2} Q'(n, u_n) - \frac{2aw^2}{u_n^3} Q(n, u_n) - \frac{aw^2}{u_n^2} Q'(n+r, u_{n+r}) = 0$$

multiply this equation by $\frac{u_n^2}{aw^2}$

$$Q'(n, u_n) - \frac{2}{u_n} Q(n, u_n) - Q'(n+r, u_{n+r}) = 0 \tag{3.3}$$

differentiate the equation with respect to u_n keeping u_{n+r} fixed

$$Q''(n, u_n) - \frac{2}{u_n} Q'(n, u_n) + \frac{2}{u_n^2} Q(n, u_n) = 0$$

again multiply by u_n^2

$$u_n^2 Q''(n,u_n) - 2u_n Q'(n,u_n) + 2Q(n,u_n) = 0$$

which is an Euler Equation, whose solution is

$$Q(n,u_n) = \alpha_n u_n^2 + \beta_n u_n$$

thus

$$Q'(n,u_n) = 2\alpha_n u_n + \beta_n$$

substitute into equation (3.3)

$$\begin{aligned} 0 &= 2\alpha_n u_n + \beta_n - 2\alpha_n u_n - 2\beta_n - 2\alpha_{n+r} u_{n+r} - \beta_{n+r} \\ &= -\beta_n - \beta_{n+r} - 2\alpha_{n+r} u_{n+r} \end{aligned}$$

comparing both sides of the last equation, we get

$$\alpha_{n+r} = 0 \text{ and so } \ \alpha_n = 0$$

we have also

$$\beta_{n+r} + \beta_n = 0 \tag{3.4}$$

which is a r^{th} order linear homogeneous difference equation. The characteristic equation of equation (3.4) is

$$\lambda^r + 1 = 0 \tag{3.5}$$

thus, the general solution of equation (3.4) is

$$\beta_n = c_1(-1)^n + c_2\lambda_1^n + c_3\lambda_2^n + ... + c_r\lambda_{r-1}^n$$

where $\{-1, \lambda_1, \dots, \lambda_{r-1}\}$ are the characteristic roots and $c_1, c_2, ..., c_r$ are constants. Suppose that $c_2 = c_3 = ... = c_r = 0$ and $c_1 = 1$, so $\beta_n = (-1)^n$, which implies

$$Q(n,u_n) = (-1)^n u_n.$$

To find the invariant,

$$\frac{du_n}{(-1)^n u_n} = \frac{du_{n+r}}{(-1)^{n+r} u_{n+r}} := \frac{dv_n}{0},$$

take $\frac{du_n}{(-1)^n u_n} = \frac{du_{n+r}}{(-1)^{n+r} u_{n+r}}$ invariants,
so

$$\ln \mid u_n \mid = (-1)^r \ln \mid u_{n+r} \mid + c = -\ln \mid u_{n+r} \mid + c, \quad \text{then} \quad c = \ln \mid u_n u_{n+r} \mid$$

where c is constant, so

$$k_1 = u_n u_{n+r} \text{ where } k_1 = e^c,$$

we also have

$$\frac{du_n}{u_n} := \frac{dv_n}{0}$$

and so

$$v_n = k, \quad \text{such that} k = f(k_1)$$

where k, k_1 are constants. Let $f(k_1) = k_1$, then

$$v_n = u_{n+r}u_n,$$

and

$$v_{n+r} = u_{n+2r}u_{n+r} = \frac{u_n u_{n+r}}{a + bu_n u_{n+r}} = \frac{v_n}{a + bv_n}.$$

so

$$\frac{1}{v_{n+r}} = \frac{a}{v_n} + b \tag{3.6}$$

let

$$z_n = \frac{1}{v_n}, \tag{3.7}$$

this substitution converts equation (3.6) to the following r^{th} order linear equation

$$z_{n+r} - az_n - b = 0 \tag{3.8}$$

so

$$z_{n+r} - az_n = b$$

3.1.1 The Case $a \neq 1$

The characteristic equation of the homogeneous equation

$$z_{n+r} - az_n = 0$$

is

$$\lambda^r - a = 0.$$

The roots of the last equation are

$$\begin{aligned}
\mu_1 &= a^{\frac{1}{r}},\\
\mu_2 &= a^{\frac{1}{r}}e^{i\frac{2\pi}{r}},\\
\mu_3 &= a^{\frac{1}{r}}e^{i\frac{4\pi}{r}},\\
&\vdots\\
\mu_{r-1} &= a^{\frac{1}{r}}e^{i\frac{2(r-2)\pi}{r}}\\
\mu_r &= a^{\frac{1}{r}}e^{i\frac{2(r-1)\pi}{r}}
\end{aligned}$$

So the soluation of the homogeneous part is

$$\begin{aligned}
z_h &= c_1\mu_1^n + \hat{c}_2\mu_2^n + \hat{c}_3\mu_3^n + \cdots + \hat{c}_{r-1}\mu_{r-1}^n + \hat{c}_r\mu_r^n\\
&= a^{\frac{n}{r}}\left(c_1 + \hat{c}_2e^{i\frac{2n\pi}{r}} + \hat{c}_3e^{i\frac{4n\pi}{r}} + \cdots + \hat{c}_{r-1}e^{i\frac{2(r-2)n\pi}{r}} + \hat{c}_re^{i\frac{2(r-1)n\pi}{r}}\right)\\
&= a^{\frac{n}{r}}(c_1 + c_2\cos\frac{2n\pi}{r} + c_3\sin\frac{2n\pi}{r} + \cdots + c_{r-1}\cos\frac{(r-1)n\pi}{r} + c_r\sin\frac{(r-1)n\pi}{r})\\
&= c_1 + \sum_{j=1}^{\frac{r-1}{2}}\left[c_{2j}\cos\frac{2jn\pi}{r} + c_{2j+1}\sin\frac{2jn\pi}{r}\right],
\end{aligned}$$

where $c_1, \hat{c}_2, \hat{c}_3, \ \ldots \ , \hat{c}_{r-1}, \hat{c}_r$ are constants and

$$c_{2j} = \hat{c}_{j+1} + \hat{c}_{r-j+1}, \quad c_{2j+1} = i(\hat{c}_{j+1} - \hat{c}_{r-j+1}),$$

for $j = 1, 2, \ldots, \frac{r-1}{2}$.

The particular solution is

$$z_p = \frac{b}{1-a}$$

thus, the solution of equation (3.8) is

$$\begin{aligned}
z_n &= z_h + z_p\\
&= a^{\frac{n}{r}}\left(c_1 + \sum_{j=1}^{\frac{r-1}{2}}\left[c_{2j}cos\frac{2jn\pi}{r} + c_{2j+1}\sin\frac{2jn\pi}{r}\right]\right) + \frac{b}{1-a}
\end{aligned}$$

Given initial values $z_0, z_1, \ldots, z_{r-1}$, the constants $c_1, c_2, \ldots, c_r$ satisfy the following system of equations

$$\begin{aligned}
\big(z_0 - \frac{b}{1-a}\big) &= c_1 + \sum_{j=1}^{\frac{r-1}{2}} c_{2j} \\
\big(z_1 - \frac{b}{1-a}\big)a^{-\frac{1}{r}} &= c_1 + \sum_{j=1}^{\frac{r-1}{2}} \Big[c_{2j}\cos\frac{2j\pi}{r} + c_{2j+1}\sin\frac{2j\pi}{r}\Big] \\
\big(z_2 - \frac{b}{1-a}\big)a^{-\frac{2}{r}} &= c_1 + \sum_{j=1}^{\frac{r-1}{2}} \Big[c_{2j}\cos\frac{4j\pi}{r} + c_{2j+1}\sin\frac{4j\pi}{r}\Big] \\
&\vdots \\
\big(z_{r-2} - \frac{b}{1-a}\big)a^{-\frac{r-2}{r}} &= c_1 + \sum_{j=1}^{\frac{r-1}{2}} \Big[c_{2j}\cos\frac{2(r-2)j\pi}{r} + c_{2j+1}\sin\frac{2(r-2)j\pi}{r}\Big] \\
\big(z_{r-1} - \frac{b}{1-a}\big)a^{-\frac{r-1}{r}} &= c_1 + \sum_{j=1}^{\frac{r-1}{2}} \Big[c_{2j}\cos\frac{2(r-1)j\pi}{r} + c_{2j+1}\sin\frac{2(r-1)j\pi}{r}\Big]
\end{aligned}$$

then

$$c_1 = \frac{\Delta_1}{\Delta}, \quad c_2 = \frac{\Delta_2}{\Delta}, \quad c_3 = \frac{\Delta_3}{\Delta}, \ldots, \; c_{r-1} = \frac{\Delta_{r-1}}{\Delta} \text{ and } c_r = \frac{\Delta_r}{\Delta}, \quad c_r \text{ constants}$$

such that

$$\Delta = \begin{vmatrix} 1 & 1 & 0 & \dots & 1 & 0 \\ 1 & \cos\frac{2\pi}{r} & \sin\frac{2\pi}{r} & \dots & \cos\frac{(r-1)\pi}{r} & \sin\frac{(r-1)\pi}{r} \\ 1 & \cos\frac{4\pi}{r} & \sin\frac{4\pi}{r} & \dots & \cos\frac{2(r-1)\pi}{r} & \sin\frac{2(r-1)\pi}{r} \\ \vdots & & & \ddots & & \vdots \\ 1 & \cos\frac{2(r-2)\pi}{r} & \sin\frac{2(r-2)\pi}{r} & \dots & \cos\frac{(r-2)(r-1)\pi}{r} & \sin\frac{(r-2)(r-1)\pi}{r} \\ 1 & \cos\frac{2(r-1)\pi}{r} & \sin\frac{2(r-1)\pi}{r} & \dots & \cos\frac{(r-1)^2\pi}{r} & \sin\frac{(r-1)^2\pi}{r} \end{vmatrix}$$

$$\Delta_1 = \begin{vmatrix} \left(z_0 - \frac{b}{1-a}\right) & 1 & 0 & \dots & 1 & 0 \\ \left(z_1 - \frac{b}{1-a}\right)a^{-\frac{1}{r}} & \cos\frac{2\pi}{r} & \sin\frac{2\pi}{r} & \dots & \cos\frac{(r-1)\pi}{r} & \sin\frac{(r-1)\pi}{r} \\ \left(z_2 - \frac{b}{1-a}\right)a^{-\frac{2}{r}} & \cos\frac{4\pi}{r} & \sin\frac{4\pi}{r} & \dots & \cos\frac{2(r-1)\pi}{r} & \sin\frac{2(r-1)\pi}{r} \\ \vdots & & & \ddots & & \vdots \\ \left(z_{r-2} - \frac{b}{1-a}\right)a^{-\frac{r-2}{r}} & \cos\frac{2(r-2)\pi}{r} & \sin\frac{2(r-2)\pi}{r} & \dots & \cos\frac{(r-2)(r-1)\pi}{r} & \sin\frac{(r-2)(r-1)\pi}{r} \\ \left(z_{r-1} - \frac{b}{1-a}\right)a^{-\frac{r-1}{r}} & \cos\frac{2(r-1)\pi}{r} & \sin\frac{2(r-1)\pi}{r} & \dots & \cos\frac{(r-1)^2\pi}{r} & \sin\frac{(r-1)^2\pi}{r} \end{vmatrix}$$

$$= \left(z_0 - \frac{b}{1-a}\right)\Delta_{01} - \left(z_1 - \frac{b}{1-a}\right)a^{-\frac{1}{r}}\Delta_{11} + \left(z_2 - \frac{b}{1-a}\right)a^{-\frac{2}{r}}\Delta_{21}$$
$$- \dots - \left(z_{r-2} - \frac{b}{1-a}\right)a^{-\frac{r-2}{r}}\Delta_{(r-2)1} + \left(z_{r-1} - \frac{b}{1-a}\right)a^{-\frac{r-1}{r}}\Delta_{(r-1)1}$$

such that Δ_{j1} is the minor of an element $(j+1, 1)$ of Δ_1, $\quad j = 0, 1, \dots, r-1$.

$$\Delta_2 = \begin{vmatrix} 1 & \left(z_0 - \frac{b}{1-a}\right) & 0 & \dots & 1 & 0 \\ 1 & \left(z_1 - \frac{b}{1-a}\right)a^{-\frac{1}{r}} & \sin\frac{2\pi}{r} & \dots & \cos\frac{(r-1)\pi}{r} & \sin\frac{(r-1)\pi}{r} \\ 1 & \left(z_2 - \frac{b}{1-a}\right)a^{-\frac{2}{r}} & \sin\frac{4\pi}{r} & \dots & \cos\frac{(r-1)\pi}{r} & \sin\frac{2(r-1)\pi}{r} \\ \vdots & & & \ddots & & \vdots \\ 1 & \left(z_{r-2} - \frac{b}{1-a}\right)a^{-\frac{r-2}{r}} & \sin\frac{2(r-2)\pi}{r} & \dots & \cos\frac{(r-2)(r-1)\pi}{r} & \sin\frac{(r-2)(r-1)\pi}{r} \\ 1 & \left(z_{r-1} - \frac{b}{1-a}\right)a^{-\frac{r-1}{r}} & \sin\frac{2(r-1)\pi}{r} & \dots & \cos\frac{(r-1)^2\pi}{r} & \sin\frac{(r-1)^2\pi}{r} \end{vmatrix}$$

$$= -\left(z_0 - \frac{b}{1-a}\right)\Delta_{02} + \left(z_1 - \frac{b}{1-a}\right)a^{-\frac{1}{r}}\Delta_{12} - \left(z_2 - \frac{b}{1-a}\right)a^{-\frac{2}{r}}\Delta_{22}$$
$$+ \cdots + \left(z_{r-2} - \frac{b}{1-a}\right)a^{-\frac{r-2}{r}}\Delta_{(r-2)2} - \left(z_{r-1} - \frac{b}{1-a}\right)a^{-\frac{r-1}{r}}\Delta_{(r-1)2}$$

such that Δ_{j2} is the minor of an element $(j+1, 2)$ of Δ_2 $\quad j = 0, 1, \dots, r-1$.

$$\Delta_3 = \begin{vmatrix} 1 & 1 & \left(z_0 - \frac{b}{1-a}\right) & \dots & 1 & 0 \\ 1 & \cos\frac{2\pi}{r} & \left(z_1 - \frac{b}{1-a}\right)a^{-\frac{1}{r}} & \dots & \cos\frac{(r-1)\pi}{r} & \sin\frac{(r-1)\pi}{r} \\ 1 & \cos\frac{4\pi}{r} & \left(z_2 - \frac{b}{1-a}\right)a^{-\frac{2}{r}} & \dots & \cos\frac{2(r-1)\pi}{r} & \sin\frac{2(r-1)\pi}{r} \\ \vdots & & & \ddots & & \vdots \\ 1 & \cos\frac{2(r-2)\pi}{r} & \left(z_{r-2} - \frac{b}{1-a}\right)a^{-\frac{(r-2)}{r}} & \dots & \cos\frac{(r-2)(r-1)\pi}{r} & \sin\frac{(r-2)(r-1)\pi}{r} \\ 1 & \cos\frac{2(r-1)\pi}{r} & \left(z_{r-1} - \frac{b}{1-a}\right)a^{-\frac{r-1}{r}} & \dots & \cos\frac{(r-1)^2\pi}{r} & \sin\frac{(r-1)^2\pi}{r} \end{vmatrix}$$

$$= \left(z_0 - \frac{b}{1-a}\right)\Delta_{03} - \left(z_1 - \frac{b}{1-a}\right)a^{-\frac{1}{r}}\Delta_{13} + \left(z_2 - \frac{b}{1-a}\right)a^{-\frac{2}{r}}\Delta_{23}$$
$$- \cdots - \left(z_{r-2} - \frac{b}{1-a}\right)a^{-\frac{r-2}{r}}\Delta_{(r-2)3} + \left(z_{r-1} - \frac{b}{1-a}\right)a^{-\frac{r-1}{r}}\Delta_{(r-1)3}$$

such that Δ_{j3} is the minor of an element $(j+1, 3)$ of Δ_3, $\quad j = 0, 1, \dots, r-1$.

$$\Delta_{r-1} = \begin{vmatrix} 1 & 1 & 0 & \dots & \left(z_0 - \frac{b}{1-a}\right) & 0 \\ 1 & \cos\frac{2\pi}{r} & \sin\frac{\pi}{r} & \dots & \left(z_1 - \frac{b}{1-a}\right)a^{-\frac{1}{r}} & \sin\frac{(r-1)\pi}{r} \\ 1 & \cos\frac{4\pi}{r} & \sin\frac{4\pi}{r} & \dots & \left(z_2 - \frac{b}{1-a}\right)a^{-\frac{2}{r}} & \sin\frac{2(r-1)\pi}{r} \\ \vdots & & & \ddots & & \vdots \\ 1 & \cos\frac{2(r-2)\pi}{r} & \sin\frac{2(r-2)\pi}{r} & \dots & \left(z_{r-2} - \frac{b}{1-a}\right)a^{-\frac{r-2}{r}} & \sin\frac{(r-2)(r-1)\pi}{r} \\ 1 & \cos\frac{2(r-1)\pi}{r} & \sin\frac{2(r-1)\pi}{r} & \dots & \left(z_{r-1} - \frac{b}{1-a}\right)a^{-\frac{r-1}{r}} & \sin\frac{(r-1)^2\pi}{r} \end{vmatrix}$$

$$= -\left(z_0 - \frac{b}{1-a}\right)\Delta_{0(r-1)} + \left(z_1 - \frac{b}{1-a}\right)a^{-\frac{1}{r}}\Delta_{1(r-1)} - \left(z_2 - \frac{b}{1-a}\right)a^{-\frac{2}{r}}\Delta_{2(r-1)} + \dots + \left(z_{r-2} - \frac{b}{1-a}\right)a^{-\frac{r-2}{r}}\Delta_{(r-2)(r-1)} - \left(z_{r-1} - \frac{b}{1-a}\right)a^{-\frac{r-1}{r}}\Delta_{(r-1)(r-1)}$$

such that $\Delta_{j(r-1)}$ is the minor of an element $(j+1,(r-1))$ of Δ_{r-1}, $\quad j = 0, 1, \dots, r-1$.

$$\Delta_r = \begin{vmatrix} 1 & 1 & 0 & \dots & 1 & \left(z_0 - \frac{b}{1-a}\right) \\ 1 & \cos\frac{2\pi}{r} & \sin\frac{2\pi}{r} & \dots & \cos\frac{(r-1)\pi}{r} & \left(z_1 - \frac{b}{1-a}\right)a^{-\frac{1}{r}} \\ 1 & \cos\frac{4\pi}{r} & \sin\frac{4\pi}{r} & \dots & \cos\frac{2(r-1)\pi}{r} & \left(z_2 - \frac{b}{1-a}\right)a^{-\frac{2}{r}} \\ \vdots & & & \ddots & & \vdots \\ 1 & \cos\frac{2(r-2)\pi}{r} & \sin\frac{2(r-2)\pi}{r} & \dots & \cos\frac{(r-2)(r-1)\pi}{r} & \left(z_{r-2} - \frac{b}{1-a}\right)a^{-\frac{(r-2)}{r}} \\ 1 & \cos\frac{2(r-1)\pi}{r} & \sin\frac{2(r-1)\pi}{r} & \dots & \cos\frac{(r-1)^2\pi}{r} & \left(z_{r-1} - \frac{b}{1-a}\right)a^{-\frac{r-1}{r}} \end{vmatrix}$$

$$= z_0\Delta_{0r} - z_1\Delta_{r1} + z_2\Delta_{2r} - \dots - z_{r-2}\Delta_{(r-2)r} + z_{r-1}\Delta_{(r-1)r} - \frac{b}{1-a}\left(\Delta_{0r} - \Delta_{1r} + \Delta_{2r} - \dots - \Delta_{(r-2)r} + \Delta_{(r-1)r}\right)$$

$$= \left(z_0 - \frac{b}{1-a}\right)\Delta_{0r} - \left(z_1 - \frac{b}{1-a}\right)a^{-\frac{1}{r}}\Delta_{1r} + \left(z_2 - \frac{b}{1-a}\right)a^{-\frac{2}{r}}\Delta_{2r} - \dots - \left(z_{r-2} - \frac{b}{1-a}\right)a^{-\frac{r-2}{r}}\Delta_{(r-2)r} - \left(z_{r-1} - \frac{b}{1-a}\right)a^{-\frac{r-1}{r}}\Delta_{(r-1)r}$$

such that Δ_{jr} is the minor of an element $(j+1, r)$ of Δ_r, $\quad j = 0, 1, \ldots, r-1$.

So

$$\begin{aligned} c_x &= \frac{\Delta_x}{\Delta} \\ &= \frac{1}{\Delta}\sum_{t=0}^{r-1}(-1)^{t+x+1}\big(z_t - \frac{b}{1-a}\big)a^{-\frac{t}{r}}\Delta_{tx} \\ &= \frac{1}{\Delta}\sum_{t=0}^{r-1}(-1)^{t+x+1}\big(\frac{1}{u_t u_{t+r}} - \frac{b}{1-a}\big)a^{-\frac{t}{r}}\Delta_{tx}\ , \quad x = 1, 2, \ldots, r \end{aligned}$$

thus,

$$\begin{aligned} c_{2j} &= \frac{1}{\Delta}\sum_{t=0}^{r-1}(-1)^{t+2j+1}\big(z_t - \frac{b}{1-a}\big)a^{-\frac{t}{r}}\Delta_{t(2j)} \\ &= \frac{1}{\Delta}\sum_{t=0}^{r-1}(-1)^{t+1}\big(\frac{1}{u_t u_{t+r}} - \frac{b}{1-a}\big)a^{-\frac{t}{r}}\Delta_{t(2j)} \end{aligned}$$

$$\begin{aligned} c_{2j+1} &= \frac{1}{\Delta}\sum_{t=0}^{r-1}(-1)^{t+2j+1+1}\big(z_t - \frac{b}{1-a}\big)a^{-\frac{t}{r}}\Delta_{t(2j+1)} \\ &= \frac{1}{\Delta}\sum_{t=0}^{r-1}(-1)^{t}\big(\frac{1}{u_t u_{t+r}} - \frac{b}{1-a}\big)a^{-\frac{t}{r}}\Delta_{t(2j+1)} \end{aligned}$$

The invariant v_n is given by

$$\begin{aligned} v_n &= \frac{1}{z_n} \\ &= \frac{1}{a^{\frac{n}{r}}\left(c_1 + \sum_{j=1}^{\frac{r-1}{2}}\left[c_{2j}\cos\frac{2jn\pi}{r} + c_{2j+1}\sin\frac{2jn\pi}{r}\right]\right) + \frac{b}{1-a}} = u_{n+r}u_n \end{aligned}$$

and so

$$u_{n+r} = \frac{1}{\left[a^{\frac{n}{r}}\left(c_1 + \sum_{j=1}^{\frac{r-1}{2}}\left[c_{2j}\cos\frac{2jn\pi}{r} + c_{2j+1}\sin\frac{2jn\pi}{r}\right]\right) + \frac{b}{1-a}\right]u_n}.$$

To solve the last equation we need to obtain the canonical coordinate,

$$s_n = \int \frac{du_n}{(-1)^n u_n}$$

$$= (-1)^n \ln |u_n|$$

so

$$\begin{aligned}
s_{n+r} - s_n &= (-1)^{n+r} \ln |u_{n+r}| - (-1)^n \ln |u_n| \\
&= (-1)^{n+1}[\ln |u_{n+r}| + \ln |u_n|] \\
&= (-1)^{n+1} \ln |u_{n+r} u_n| \\
&= (-1)^{n+1} \ln |v_n| \\
&= (-1)^{n+1} \ln \left| \frac{1}{a^{\frac{n}{r}} \left(c_1 + \sum_{j=1}^{\frac{r-1}{2}} \left[c_{2j} \cos \frac{2jn\pi}{r} + c_{2j+1} \sin \frac{2jn\pi}{r}\right]\right) + \frac{b}{1-a}} \right| \\
&= (-1)^{n+2} \ln \left| a^{\frac{n}{r}} \left(c_1 + \sum_{j=1}^{\frac{r-1}{2}} \left[c_{2j} \cos \frac{2jn\pi}{r} + c_{2j+1} \sin \frac{2jn\pi}{r}\right]\right) + \frac{b}{1-a} \right|
\end{aligned}$$

and so

$$s_{n+r} - s_n = (-1)^n \ln \left| a^{\frac{n}{r}} \left(c_1 + \sum_{j=1}^{\frac{r-1}{2}} \left[c_{2j} \cos \frac{2jn\pi}{r} + c_{2j+1} \sin \frac{2jn\pi}{r}\right]\right) + \frac{b}{1-a} \right|$$

which is a r^{th} order non homogeneous difference equation that can be solved recursively. Let $s_0, s_1, \ldots, s_{r-1}$ be given and let

$$f(n) = (-1)^n \ln \left| a^{\frac{n}{r}} \left(c_1 + \sum_{j=1}^{\frac{r-1}{2}} \left[c_{2j} \cos \frac{2jn\pi}{r} + c_{2j+1} \sin \frac{2jn\pi}{r}\right]\right) + \frac{b}{1-a} \right|$$

then

$$s_{n+r} = s_n + f(n)$$

$$\begin{aligned}
s_r &= s_0 + f(0) \\
s_{2r} &= s_r + f(r) = s_0 + f(0) + f(r) \\
s_{3r} &= s_{2r} + f(2r) = s_0 + f(0) + f(r) + f(2r) \\
s_{4r} &= s_{3r} + f(3r) = s_0 + f(0) + f(r) + f(2r) + f(3r) \\
&\vdots
\end{aligned}$$

so, for $n = mr$, $m = 1, 2, 3, ...$

$$\begin{aligned} s_n = s_{mr} &= s_0 + f(0) + f(r) + f(2r) + f(3r) + ... + f\big((m-1)r\big) \\ &= s_0 + \sum_{k=0}^{m-1} f(kr) \end{aligned}$$

but

$$\begin{aligned} f(kr) &= (-1)^{kr} \ln \left| a^k \big(c_1 + \sum_{j=1}^{\frac{r-1}{2}} \big[c_{2j} \cos \frac{2j(kr)\pi}{r} + c_{2j+1} \sin \frac{2j(kr)\pi}{r}\big]\big) + \frac{b}{1-a} \right| \\ &= (-1)^{kr} \ln \left| a^k \big(c_1 + \sum_{j=1}^{\frac{r-1}{2}} c_{2j}\big) + \frac{b}{1-a} \right| \end{aligned}$$

hence,

$$s_{mr} = s_0 + \sum_{k=0}^{m-1} (-1)^{rk} \ln \left| a^k \big(c_1 + \sum_{j=1}^{\frac{r-1}{2}} c_{2j}\big) + \frac{b}{1-a} \right|. \qquad (3.9)$$

Also

$$\begin{aligned} s_{r+1} &= s_1 + f(1) \\ s_{2r+1} &= s_{r+1} + f(r+1) = s_1 + f(1) + f(r+1) \\ s_{3r+1} &= s_{2r+1} + f(2r+1) = s_1 + f(1) + f(r+1) + f(2r+1) \\ s_{4r+1} &= s_{3r+1} + f(3r+1) = s_1 + f(1) + f(r+1) + f(2r+1) + f(3r+1) \\ &\vdots \end{aligned}$$

so, for $n = mr + 1$, $m = 1, 2, 3, ...$

$$\begin{aligned} s_n = s_{mr+1} &= s_1 + f(1) + f(r+1) + f(2r+1) + \cdots + f\big((m-1)r+1\big) \\ &= s_1 + \sum_{k=0}^{m-1} f(kr+1) \end{aligned}$$

but

$$\begin{aligned} f(rk+1) &= (-1)^{rk+1} \ln \left| a^{\frac{rk+1}{r}} \big(c_1 + \sum_{j=1}^{\frac{r-1}{2}} \big[c_{2j} \cos \frac{2j(rk+1)\pi}{r} + c_{2j+1} \sin \frac{2j(rk+1)\pi}{r} \big] \big) + \frac{b}{1-a} \right| \\ &= (-1)^{rk+1} \ln \left| a^{k+\frac{1}{r}} \big(c_1 + \sum_{j=1}^{\frac{r-1}{2}} \big[c_{2j} \cos \frac{2j\pi}{r} + c_{2j+1} \sin \frac{2j\pi}{r} \big] \big) + \frac{b}{1-a} \right| \end{aligned}$$

hence,

$$s_{mr+1} = s_1 + \sum_{k=0}^{m-1} (-1)^{rk+1} \ln \left| a^{k+\frac{1}{r}} \big(c_1 + \sum_{j=1}^{\frac{r-1}{2}} \big[c_{2j} \cos \frac{2j\pi}{r} + c_{2j+1} \sin \frac{2j\pi}{r} \big] \big) + \frac{b}{1-a} \right|. \tag{3.10}$$

Also

$$\begin{aligned} s_{r+2} &= s_2 + f(2) \\ s_{2r+2} &= s_{r+2} + f(r+2) = s_2 + f(2) + f(r+2) \\ s_{3r+2} &= s_{2r+2} + f(2r+2) = s_2 + f(2) + f(r+2) + f(2r+2) \\ s_{4r+2} &= s_{3r+2} + f(3r+2) = s_2 + f(2) + f(r+2) + f(2r+2) + f(3r+2) \\ &\vdots \end{aligned}$$

so, for $n = mr+2,\ m = 1, 2, 3, ...$

$$\begin{aligned} s_n = s_{mr+2} &= s_2 + f(2) + f(r+2) + f(2r+2) + \cdots + f\big((m-1)r+2\big) \\ &= s_2 + \sum_{k=0}^{m-1} f(kr+2) \end{aligned}$$

but

$$\begin{aligned} f(kr+2) &= (-1)^{kr+2} \ln \left| a^{\frac{kr+2}{r}} \Big(c_1 + \sum_{j=1}^{\frac{r-1}{2}} \big[c_{2j} \cos \frac{2j(kr+2)\pi}{r} + c_{2j+1} \sin \frac{2j(kr+2)\pi}{r} \big] \Big) + \frac{b}{1-a} \right| \\ &= (-1)^{kr+2} \ln \left| a^{k+\frac{2}{r}} \Big(c_1 + \sum_{j=1}^{\frac{r-1}{2}} \big[c_{2j} \cos \frac{4j\pi}{r} + c_{2j+1} \sin \frac{4j\pi}{r} \big] \Big) + \frac{b}{1-a} \right| \end{aligned}$$

hence,

$$s_{mr+2} = s_2 + \sum_{k=0}^{m-1}(-1)^{kr+2}\ln\left|a^{k+\frac{2}{r}}\left(c_1 + \sum_{j=1}^{\frac{r-1}{2}}\left[c_{2j}\cos\frac{4j\pi}{r} + c_{2j+1}\sin\frac{4j\pi}{r}\right]\right) + \frac{b}{1-a}\right|. \tag{3.11}$$

Recursively up to $n = mr + (r-1), \quad m = 1, 2, \ldots,$ we get

$$s_n = s_{mr+(r-1)} = s_2 + \sum_{k=0}^{m-1} f\big(kr + (r-1)\big)$$

and

$$\begin{aligned} f\big(kr+(r-1)\big) &= (-1)^{kr+(r-1)}\ln\Bigg|a^{\frac{kr+(r-1)}{r}}\Bigg(c_1 + \sum_{j=1}^{\frac{r-1}{2}}\Big[c_{2j}\cos\frac{2j\big(kr+(r-1)\big)\pi}{r} \\ &\qquad + c_{2j+1}\sin\frac{2j\big(kr+(r-1)\big)\pi}{r}\Big]\Bigg) + \frac{b}{1-a}\Bigg| \\ &= (-1)^{kr+(r-1)}\ln\Bigg|a^{k+\frac{r-1}{r}}\big(c_1 + \sum_{j=1}^{\frac{r-1}{2}}\big[c_{2j}\cos\frac{2(r-1)j\pi}{r} \\ &\qquad + c_{2j+1}\sin\frac{2(r-1)j\pi}{r}\big]\big) + \frac{b}{1-a}\Bigg| \end{aligned}$$

hence,

$$\begin{aligned} s_{mr+(r-1)} &= s_{(r-1)} + \sum_{k=0}^{m-1}(-1)^{kr+(r-1)}\ln\Bigg|a^{k+\frac{r-1}{r}}\big(c_1 + \sum_{j=1}^{\frac{r-1}{2}}\big[c_{2j}\cos\frac{2(r-1)j\pi}{r} \\ &\qquad + c_{2j+1}\sin\frac{2(r-1)j\pi}{r}\big]\big) + \frac{b}{1-a}\Bigg|. \end{aligned} \tag{3.12}$$

Now, from (3.9,3.10,3.11,3.12) we obtain, for $n = mr + l, \quad m = 1, 2, \ldots$ and $l = 0, 1, 2, \ldots, r-1$

$$\begin{aligned} s_n = s_{mr+l} &= s_l + \sum_{k=0}^{m-1}(-1)^{kr+l}\ln\Bigg|a^{k+\frac{l}{r}}\big(c_1 + \sum_{j=1}^{\frac{r-1}{2}}\big[c_{2j}\cos\frac{2lj\pi}{r} \\ &\qquad + c_{2j+1}\sin\frac{2lj\pi}{r}\big]\big) + \frac{b}{1-a}\Bigg|. \end{aligned}$$

The canonical coordinate

$$s_{mr+l} = (-1)^{mr+l} \ln |u_{mr+l}|, \quad m = 1, 2, 3, \dots \quad \text{and} \quad l = 1, 2, \dots, (r-1)$$

which implies

$$\begin{aligned} u_{mr+l} &= \exp\left((-1)^{mr+l} s_{mr+l}\right) \\ &= u_l^{(-1)^{mr}} \prod_{k=0}^{m-1} \left(a^{k+\frac{l}{r}} \big(c_1 + \sum_{j=1}^{\frac{r-1}{2}} \big[c_{2j} \cos \frac{2lj\pi}{r} + c_{2j+1} \sin \frac{2lj\pi}{r}\big]\big) + \frac{b}{1-a} \right)^{(-1)^{r(k+m)}} \quad (3.13) \end{aligned}$$

Now, to determine the forbidden set. From(3.20), $n = mr + l$

$$u_n = u_{mr+l} = u_l^{(-1)^{mr}} \prod_{k=0}^{m-1} \left(a^{k+\frac{l}{r}} \big(c_1 + \sum_{j=1}^{\frac{r-1}{2}} \big[c_{2j} \cos \frac{2lj\pi}{r} + c_{2j+1} \sin \frac{2lj\pi}{r}\big]\big) + \frac{b}{1-a} \right)^{(-1)^{r(k+m)}}$$

let

$$f_l(m,k) = \left(a^{k+\frac{l}{r}} \big(c_1 + \sum_{j=1}^{\frac{r-1}{2}} \big[c_{2j} \cos \frac{2lj\pi}{r} + c_{2j+1} \sin \frac{2lj\pi}{r}\big]\big) + \frac{b}{1-a} \right)^{(-1)^{r(k+m)}}$$

then

$$\begin{aligned}
f_l(m,k) &= \Bigg(a^{k+\frac{l}{r}}\Bigg(\Bigg[\frac{1}{\Delta}\sum_{t=0}^{r-1}(-1)^t(\frac{1}{u_tu_{t+r}}-\frac{b}{1-a})a^{-\frac{t}{r}}\Delta_{t1}\Bigg] \\
&\quad +\sum_{j=1}^{\frac{r-1}{2}}\Bigg[\frac{1}{\Delta}\sum_{t=0}^{r-1}(-1)^{t+1}(\frac{1}{u_tu_{t+r}}-\frac{b}{1-a})a^{-\frac{t}{r}}\Delta_{t(2j)}\Bigg]\cos\frac{2jl\pi}{r} \\
&\quad +\sum_{j=1}^{\frac{r-1}{2}}\Bigg[\frac{1}{\Delta}\sum_{t=0}^{r-1}(-1)^t(\frac{1}{u_tu_{t+r}}-\frac{b}{1-a})a^{-\frac{t}{r}}\Delta_{t(2j+1)}\Bigg]\sin\frac{2jl\pi}{r}\Bigg)+\frac{b}{1-a}\Bigg)^{(-1)^{3(k+m)}} \\
&= \Bigg(\sum_{t=0}^{r-1}\frac{1}{u_tu_{t+r}}a^{k+\frac{l-t}{r}}\Bigg(\frac{1}{\Delta}(-1)^t\Delta_{t1}+\sum_{j=1}^{\frac{r-1}{2}}\frac{1}{\Delta}(-1)^{t+1}\Delta_{t(2j)}\cos\frac{2jl\pi}{r} \\
&\qquad +\sum_{j=1}^{\frac{r-1}{2}}\frac{1}{\Delta}(-1)^t\Delta_{t(2j+1)}\ \sin\frac{2jl\pi}{r}\Bigg) \\
&\quad -\frac{b}{1-a}\sum_{t=0}^{r-1}\Bigg(a^{k+\frac{l-t}{r}}\frac{1}{\Delta}(-1)^t\Delta_{t1}+\ a^{k+\frac{l-t}{r}}\sum_{j=1}^{\frac{r-1}{2}}\frac{1}{\Delta}(-1)^{t+1}\Delta_{t(2j)}\cos\frac{2jl\pi}{r} \\
&\qquad +\ a^{k+\frac{l-t}{r}}\sum_{j=1}^{\frac{r-1}{2}}\frac{1}{\Delta}(-1)^t\Delta_{t(2j+1)}\ \sin\frac{2jl\pi}{r}-1\Bigg)\Bigg)^{(-1)^{3(k+m)}} \qquad (3.14)
\end{aligned}$$

$$
\begin{aligned}
= \quad & \Bigg(\sum_{t=0}^{r-1} \frac{u_0 u_1 u_2 \ldots u_{2r-1}(1-a)/u_t u_{t+r}}{u_0 u_1 u_2 \ldots u_{2r-1}(1-a)} a^{k+\frac{l-t}{r}} \Bigg(\frac{1}{\Delta}(-1)^t \Delta_{t1} \\
& + \sum_{j=1}^{\frac{r-1}{2}} \frac{1}{\Delta}(-1)^{t+1} \Delta_{t(2j)} \cos \frac{2jl\pi}{r} \\
& + \sum_{j=1}^{\frac{r-1}{2}} \frac{1}{\Delta}(-1)^{t} \Delta_{t(2j+1)} \ \sin \frac{2jl\pi}{r} \Bigg) \\
& - \frac{u_0 u_1 u_2 \ldots u_{2r-1} b}{u_0 u_1 u_2 \ldots u_{2r-1}(1-a)} \sum_{t=0}^{r-1} \Bigg(a^{k+\frac{l-t}{r}} \frac{1}{\Delta}(-1)^t \Delta_{t1} \\
& + a^{k+\frac{l-t}{r}} \sum_{j=1}^{\frac{r-1}{2}} \frac{1}{\Delta}(-1)^{t+1} \Delta_{t(2j)} \cos \frac{2jl\pi}{r} \\
& + a^{k+\frac{l-t}{r}} \sum_{j=0}^{\frac{r-1}{2}} \frac{1}{\Delta}(-1)^{t} \Delta_{t(2j+1)} \ \sin \frac{2jl\pi}{r} - 1 \Bigg) \Bigg)^{(-1)^{3(k+m)}} \qquad (3.15)
\end{aligned}
$$

Now, we have the following

1. if $k+m =$ even number for some m and k, from (3.15) we have $f_l(m,k)$ is defined where

$$u_0 u_1 u_2 \ldots u_{2r-1} \neq 0$$

2. if $k+m =$ odd number for some m and k, from (3.14) we have $f_l(m,k)$ is undefined

when

$$
\begin{aligned}
0 &= \sum_{t=0}^{r-1} \frac{1}{u_t u_{t+r}} a^{k+\frac{l-t}{r}} \Bigg(\frac{1}{\Delta}(-1)^t \Delta_{1t} + \sum_{j=1}^{\frac{r-1}{2}} \frac{1}{\Delta}(-1)^{t+1}\Delta_{(2j)t} \cos\frac{2jl\pi}{r} \\
&\qquad\qquad + \sum_{j=0}^{\frac{r-1}{2}} \frac{1}{\Delta}(-1)^t \Delta_{(2j+1)t} \ \sin\frac{2jl\pi}{r} \Bigg) \\
&\quad - \frac{b}{1-a} \sum_{t=0}^{r-1} \Bigg(a^{k+\frac{l-t}{r}} \frac{1}{\Delta}(-1)^t \Delta_{1t} + \ a^{k+\frac{l-t}{r}} \sum_{j=1}^{\frac{r-1}{2}} \frac{1}{\Delta}(-1)^{t+1}\Delta_{(2j)t} \cos\frac{2jl\pi}{r} \\
&\qquad\qquad + a^{k+\frac{l-t}{r}} \sum_{j=0}^{\frac{r-1}{2}} \frac{1}{\Delta}(-1)^t \Delta_{(2j+1)t} \ \sin\frac{2jl\pi}{r} - 1 \Bigg) \\
&= \sum_{t=0}^{r-1} \frac{1}{u_t u_{t+r}} \xi_{tk} - \frac{b}{1-a} \xi_k
\end{aligned}
$$

where

$$
\begin{aligned}
\xi_{tk}^l &= a^{k+\frac{l-t}{r}} \Bigg(\frac{1}{\Delta}(-1)^t \Delta_{1t} + \sum_{j=1}^{\frac{r-1}{2}} \frac{1}{\Delta}(-1)^{t+2j+1}\Delta_{(2j)t} \cos\frac{2jl\pi}{r} \\
&\qquad\qquad + \sum_{j=0}^{\frac{r-1}{2}} \frac{1}{\Delta}(-1)^{t+2j} \Delta_{(2j+1)t} \ \sin\frac{2jl\pi}{r} \Bigg) \\
\xi_k^l &= \sum_{t=0}^{r-1} \Bigg(a^{k+\frac{l-t}{r}} \frac{1}{\Delta}(-1)^t \Delta_{1t} + \ a^{k+\frac{l-t}{r}} \sum_{j=1}^{\frac{r-1}{2}} \frac{1}{\Delta}(-1)^{t+2j+1}\Delta_{(2j)t} \cos\frac{2jl\pi}{r} \\
&\qquad\qquad + a^{k+\frac{l-t}{r}} \sum_{j=0}^{\frac{r-1}{2}} \frac{1}{\Delta}(-1)^{t+2j} \Delta_{(2j+1)t} \ \sin\frac{2jl\pi}{r} - 1 \Bigg)
\end{aligned}
$$

Theorem 3.1.1. *let $u_0, u_1, u_2, \ldots, u_{2r-1}$ be given real number such that $u_0 u_1 u_2 \ldots u_{2r-1} \neq 0$ and let $a \neq 1$. The forbidden set $\mathcal{F}$ of the difference equation (3.1) is given by $\mathcal{F} = \mathcal{F}^0 \bigcup \mathcal{F}^1 \bigcup, \ldots, \mathcal{F}^{r-1}$ where*

$$
\mathcal{F}^l = \bigcup_{k=1}^{\infty} \Bigg\{ \sum_{t=0}^{r-1} \Big(\frac{1}{u_t u_{t+r}} \xi_{tk}^l - \frac{b}{1-a} \xi_k^l \Big) = 0 \Bigg\}, \quad l = 0, 1, \ldots, r-1
$$

3.1.2 The Case $a = 1$

The characteristic equation is

$$\lambda^r - a = 0.$$

So the characteristic roots are

$$\begin{aligned} \lambda_1 &= 1, \\ \lambda_2 &= e^{i\frac{2\pi}{r}}, \\ \lambda_3 &= e^{i\frac{4\pi}{r}}, \\ &\vdots \\ \lambda_{r-1} &= e^{i\frac{2(r-2)\pi}{r}} \\ \lambda_r &= e^{i\frac{2(r-1)\pi}{r}} \end{aligned}$$

So the soluation of the homogeneous equation is

$$z_h = c_1 + \sum_{j=1}^{\frac{r-1}{2}} \left[c_{2j} \cos \frac{2jn\pi}{r} + c_{2j+1} \sin \frac{2jn\pi}{r} \right]$$

where c_1, c_{2j} and c_{2j+1} are constants, $j = 1, 2, \ldots, \frac{r-1}{2}$. The particular solution is

$$z_p = \frac{b}{r} n$$

thus, the solution of equation (3.8) is

$$\begin{aligned} z_n &= z_h + z_p \\ &= c_1 + \sum_{j=1}^{\frac{r-1}{2}} \left[c_{2j} \cos \frac{2jn\pi}{r} + c_{2j+1} \sin \frac{2jn\pi}{r} \right] + \frac{b}{r} n \end{aligned}$$

Given initial values $z_0, z_1, \ldots, z_{r-1}$, the constants $c_1, c_2, \ldots, c_r$ satisfy the following system of equations

$$\begin{aligned}
z_0 &= c_1 + \sum_{j=1}^{\frac{r-1}{2}} c_{2j} \\
z_1 - \frac{b}{r} &= c_1 + \sum_{j=1}^{\frac{r-1}{2}} \left[c_{2j} \cos \frac{2j\pi}{r} + c_{2j+1} \sin \frac{2j\pi}{r} \right] \\
z_2 - \frac{2b}{r} &= c_1 + \sum_{j=1}^{\frac{r-1}{2}} \left[c_{2j} \cos \frac{4j\pi}{r} + c_{2j+1} \sin \frac{4j\pi}{r} \right] \\
&\vdots \\
z_{r-2} - \frac{(r-2)b}{r} &= c_1 + \sum_{j=1}^{\frac{r-1}{2}} \left[c_{2j} \cos \frac{2(r-2)j\pi}{r} + c_{2j+1} \sin \frac{2(r-2)j\pi}{r} \right] \\
z_{r-1} - \frac{(r-1)b}{r} &= c_1 + \sum_{j=1}^{\frac{r-1}{2}} \left[c_{2j} \cos \frac{2(r-1)j\pi}{r} + c_{2j+1} \sin \frac{2(r-1)j\pi}{r} \right]
\end{aligned}$$

then

$$c_1 = \frac{\Delta_1}{\Delta}, \quad c_2 = \frac{\Delta_2}{\Delta}, \quad c_3 = \frac{\Delta_3}{\Delta}, \ldots, \; c_{r-1} = \frac{\Delta_{r-1}}{\Delta} \text{ and } c_r = \frac{\Delta_r}{\Delta}$$

such that

$$\Delta = \begin{vmatrix} 1 & 1 & 0 & \dots & 1 & 0 \\ 1 & \cos\frac{2\pi}{r} & \sin\frac{2\pi}{r} & \dots & \cos\frac{(r-1)\pi}{r} & \sin\frac{(r-1)\pi}{r} \\ 1 & \cos\frac{4\pi}{r} & \sin\frac{4\pi}{r} & \dots & \cos\frac{2(r-1)\pi}{r} & \sin\frac{2(r-1)\pi}{r} \\ \vdots & & & \ddots & & \vdots \\ 1 & \cos\frac{2(r-2)\pi}{r} & \sin\frac{2(r-2)\pi}{r} & \dots & \cos\frac{(r-2)(r-1)\pi}{r} & \sin\frac{(r-2)(r-1)\pi}{r} \\ 1 & \cos\frac{2(r-1)\pi}{r} & \sin\frac{2(r-1)\pi}{r} & \dots & \cos\frac{(r-1)^2\pi}{r} & \sin\frac{(r-1)^2\pi}{r} \end{vmatrix}$$

$$\Delta_1 = \begin{vmatrix} z_0 & 1 & 0 & \dots & 1 & 0 \\ z_1 - \frac{b}{r} & \cos\frac{2\pi}{r} & \sin\frac{2\pi}{r} & \dots & \cos\frac{(r-1)\pi}{r} & \sin\frac{(r-1)\pi}{r} \\ z_2 - \frac{2b}{r} & \cos\frac{4\pi}{r} & \sin\frac{4\pi}{r} & \dots & \cos\frac{2(r-1)\pi}{r} & \sin\frac{2(r-1)\pi}{r} \\ \vdots & & & \ddots & & \vdots \\ z_{r-2} - \frac{(r-2)b}{r} & \cos\frac{2(r-2)\pi}{r} & \sin\frac{2(r-2)\pi}{r} & \dots & \cos\frac{(r-2)(r-1)\pi}{r} & \sin\frac{(r-2)(r-1)\pi}{r} \\ z_{r-1} - \frac{(r-1)b}{r} & \cos\frac{2(r-1)\pi}{r} & \sin\frac{2(r-1)\pi}{r} & \dots & \cos\frac{(r-1)^2\pi}{r} & \sin\frac{(r-1)^2\pi}{r} \end{vmatrix}$$

$$= z_0\Delta_{01} - \left(z_1 - \frac{b}{r}\right)\Delta_{11} + \left(z_2 - \frac{2b}{r}\right)\Delta_{21} - \cdots - \left(z_{r-2} - \frac{(r-2)b}{r}\right)\Delta_{(r-2)1} + \left(z_{r-1} - \frac{(r-1)b}{r}\right)\Delta_{(r-1)1}$$

such that Δ_{j1} is the minor of an element $(j+1,1)$ of Δ_1, $\quad j = 0, 1, \dots, r-1$.

$$\Delta_2 = \begin{vmatrix} 1 & z_0 & 0 & \dots & 1 & 0 \\ 1 & z_1 - \frac{b}{r} & \sin\frac{2\pi}{r} & \dots & \cos\frac{(r-1)\pi}{r} & \sin\frac{(r-1)\pi}{r} \\ 1 & z_2 - \frac{2b}{r} & \sin\frac{4\pi}{r} & \dots & \cos\frac{2(r-1)\pi}{r} & \sin\frac{2(r-1)\pi}{r} \\ \vdots & & & \ddots & & \vdots \\ 1 & z_{r-2} - \frac{(r-2)b}{r} & \sin\frac{2(r-2)\pi}{r} & \dots & \cos\frac{(r-2)(r-1)\pi}{r} & \sin\frac{(r-2)(r-1)\pi}{r} \\ 1 & z_{r-1} - \frac{(r-1)b}{r} & \sin\frac{2(r-1)\pi}{r} & \dots & \cos\frac{(r-1)^2\pi}{r} & \sin\frac{(r-1)^2\pi}{r} \end{vmatrix}$$

$$= -z_0\Delta_{02} + \left(z_1 - \frac{b}{r}\right)\Delta_{12} - \left(z_2 - \frac{2b}{r}\right)\Delta_{22} + \cdots + \left(z_{r-2} - \frac{(r-2)b}{r}\right)\Delta_{(r-2)2} - \left(z_{r-1} - \frac{(r-1)b}{r}\right)\Delta_{(r-1)2}$$

such that Δ_{j2} is the minor of an element $(j+1,2)$ of $\Delta_2 \quad j=0,1,\ldots,r-1$.

$$\Delta_3 = \begin{vmatrix} 1 & 1 & z_0 & \ldots & 1 & 0 \\ 1 & \cos\frac{2\pi}{r} & z_1-\frac{b}{r} & \ldots & \cos\frac{(r-1)\pi}{r} & \sin\frac{(r-1)\pi}{r} \\ 1 & \cos\frac{4\pi}{r} & z_2-\frac{2b}{r} & \ldots & \cos\frac{2(r-1)\pi}{r} & \sin\frac{2(r-1)\pi}{r} \\ \vdots & & & \ddots & & \vdots \\ 1 & \cos\frac{2(r-2)\pi}{r} & z_{r-2}-\frac{(r-2)b}{r} & \ldots & \cos\frac{(r-2)(r-1)\pi}{r} & \sin\frac{(r-2)(r-1)\pi}{r} \\ 1 & \cos\frac{2(r-1)\pi}{r} & z_{r-1}-\frac{(r-1)b}{r}a^{-\frac{r-1}{r}} & \ldots & \cos\frac{(r-1)^2\pi}{r} & \sin\frac{(r-1)^2\pi}{r} \end{vmatrix}$$

$$= z_0\Delta_{03} - \left(z_1-\frac{b}{r}\right)\Delta_{13} + \left(z_2-\frac{2b}{r}\right)\Delta_{23}$$
$$- \cdots - \left(z_{r-2}-\frac{(r-2)b}{r}\right)\Delta_{(r-2)3} + \left(z_{r-1}-\frac{(r-1)b}{r}\right)\Delta_{(r-1)3}$$

such that Δ_{j3} is the minor of an element $(j+1,3)$ of $\Delta_3, \quad j=0,1,\ldots,r-1$.

$$\Delta_{r-1} = \begin{vmatrix} 1 & 1 & 0 & \ldots & z_0 & 0 \\ 1 & \cos\frac{2\pi}{r} & \sin\frac{2\pi}{r} & \ldots & z_1-\frac{b}{r} & \sin\frac{(r-1)\pi}{r} \\ 1 & \cos\frac{4\pi}{r} & \sin\frac{4\pi}{r} & \ldots & z_2-\frac{2b}{r} & \sin\frac{2(r-1)\pi}{r} \\ \vdots & & & \ddots & & \vdots \\ 1 & \cos\frac{2(r-2)\pi}{r} & \sin\frac{2(r-2)\pi}{r} & \ldots & z_{r-2}-\frac{(r-2)b}{r} & \sin\frac{(r-2)(r-1)\pi}{r} \\ 1 & \cos\frac{2(r-1)\pi}{r} & \sin\frac{2(r-1)\pi}{r} & \ldots & z_{r-1}-\frac{(r-1)b}{r} & \sin\frac{(r-1)^2\pi}{r} \end{vmatrix}$$

$$= -z_0\Delta_{0(r-1)} + \left(z_1-\frac{b}{r}\right)\Delta_{1(r-1)} - \left(z_2-\frac{2b}{r}\right)\Delta_{2(r-1)}$$
$$+ \cdots + \left(z_{r-2}-\frac{(r-2)b}{r}\right)\Delta_{(r-2)(r-1)} - \left(z_{r-1}-\frac{(r-1)b}{r}\right)\Delta_{(r-1)(r-1)}$$

such that $\Delta_{j(r-1)}$ is the minor of an element $(j+1,(r-1))$ of Δ_{r-1}, $\quad j = 0, 1, \ldots, r-1$.

$$\Delta_r = \begin{vmatrix} 1 & 1 & 0 & \ldots & 1 & z_0 \\ 1 & \cos\frac{2\pi}{r} & \sin\frac{2\pi}{r} & \ldots & \cos\frac{(r-1)\pi}{r} & z_1 - \frac{b}{r} \\ 1 & \cos\frac{4\pi}{r} & \sin\frac{4\pi}{r} & \ldots & \cos\frac{2(r-1)\pi}{r} & z_2 - \frac{2b}{r} \\ \vdots & & & \ddots & & \vdots \\ 1 & \cos\frac{2(r-2)\pi}{r} & \sin\frac{2(r-2)\pi}{r} & \ldots & \cos\frac{(r-2)(r-1)\pi}{r} & z_{r-2} - \frac{(r-2)b}{r} \\ 1 & \cos\frac{2(r-1)\pi}{r} & \sin\frac{2(r-1)\pi}{r} & \ldots & \cos\frac{(r-1)^2\pi}{r} & z_{r-1} - \frac{(r-1)b}{r} \end{vmatrix}$$

$$= z_0\Delta_{0r} - \Big(z_1 - \frac{b}{r}\Big)\Delta_{1r} + \Big(z_2 - \frac{2b}{r}\Big)\Delta_{2r} - \cdots - \Big(z_{r-2} - \frac{(r-2)b}{r}\Big)\Delta_{(r-2)r} - \Big(z_{r-1} - \frac{(r-1)b}{r}\Big)\Delta_{(r-1)r}$$

such that Δ_{jr} is the minor of an element $(j+1, r)$ of Δ_r, $\quad j = 0, 1, \ldots, r-1$.

So

$$\begin{aligned} c_x &= \frac{\Delta_x}{\Delta} \\ &= \frac{1}{\Delta}\sum_{t=0}^{r-1}(-1)^{t+x+1}\big(\frac{1}{u_t u_{t+r}} - \frac{tb}{r}\big)\Delta_{tx}\ , \quad x = 1, 2, \ldots, r \end{aligned}$$

thus,

$$\begin{aligned} c_{2j} &= \frac{1}{\Delta}\sum_{t=0}^{r-1}(-1)^{t+1}\big(\frac{1}{u_t u_{t+r}} - \frac{tb}{r}\big)\Delta_{t(2j)} \\ c_{2j+1} &= \frac{1}{\Delta}\sum_{t=0}^{r-1}(-1)^{t}\big(\frac{1}{u_t u_{t+r}} - \frac{tb}{r}\big)\Delta_{t(2j+1)} \end{aligned}$$

The invariant v_n is given by

$$\begin{aligned} v_n &= \frac{1}{z_n} \\ &= \frac{1}{c_1 + \sum_{j=1}^{\frac{r-1}{2}}\left[c_{2j}\cos\frac{2jn\pi}{r} + c_{2j+1}\sin\frac{2jn\pi}{r}\right] + \frac{b}{r}n} = u_{n+r}u_n \end{aligned}$$

and so

$$u_{n+r} = \frac{1}{\left(c_1 + \sum_{j=1}^{\frac{r-1}{2}} \left[c_{2j}\cos\frac{2jn\pi}{r} + c_{2j+1}\sin\frac{2jn\pi}{r}\right] + \frac{b}{r}n\right)u_n}.$$

Using the canonical coordinate,

$$s_n = \int \frac{du_n}{(-1)^n u_n}$$

$$= (-1)^n \ln|u_n|$$

so

$$s_{n+r} - s_n = (-1)^n \ln\left|c_1 + \sum_{j=1}^{\frac{r-1}{2}} \left[c_{2j}\cos\frac{2jn\pi}{r} + c_{2j+1}\sin\frac{2jn\pi}{r}\right] + \frac{b}{r}n\right|$$

which is a r^{th} order non homogeneous difference equation that can be solved recursively. Let $s_0, s_1, \ldots, s_{r-1}$ be given and let

$$f(n) = (-1)^n \ln\left|c_1 + \sum_{j=1}^{\frac{r-1}{2}} \left[c_{2j}\cos\frac{2jn\pi}{r} + c_{2j+1}\sin\frac{2jn\pi}{r}\right] + \frac{b}{r}n\right|$$

then

$$s_{n+r} = s_n + f(n)$$

$$\begin{aligned}
s_r &= s_0 + f(0) \\
s_{2r} &= s_r + f(r) = s_0 + f(0) + f(r) \\
s_{3r} &= s_{2r} + f(2r) = s_0 + f(0) + f(r) + f(2r) \\
s_{4r} &= s_{3r} + f(3r) = s_0 + f(0) + f(r) + f(2r) + f(3r) \\
&\vdots
\end{aligned}$$

so, for $n = mr, \qquad m = 1, 2, 3, \ldots$

$$\begin{aligned}
s_n = s_{mr} &= s_0 + f(0) + f(r) + f(2r) + f(3r) + \ldots + f\big((m-1)r\big) \\
&= s_0 + \sum_{k=0}^{m-1} f(kr)
\end{aligned}$$

but

$$\begin{aligned} f(kr) &= (-1)^{kr}\ln\left|c_1+\sum_{j=1}^{\frac{r-1}{2}}\big[c_{2j}\cos\frac{2jkr\pi}{r}+c_{2j+1}\sin\frac{2jkr\pi}{r}\big]+\frac{b}{r}kr\right| \\ &= (-1)^{kr}\ln\left|a^k\big(c_1+\sum_{j=1}^{\frac{r-1}{2}}c_{2j}\big)+bk\right| \end{aligned}$$

hence,

$$s_{mr}=s_0+\sum_{k=0}^{m-1}(-1)^{kr}\ln\left|a^k\big(c_1+\sum_{j=1}^{\frac{r-1}{2}}c_{2j}\big)+bk\right|. \tag{3.16}$$

Also

$$\begin{aligned} s_{r+1} &= s_1+f(1) \\ s_{2r+1} &= s_{r+1}+f(r+1)=s_1+f(1)+f(r+1) \\ s_{3r+1} &= s_{2r+1}+f(2r+1)=s_1+f(1)+f(r+1)+f(2r+1) \\ s_{4r+1} &= s_{3r+1}+f(3r+1)=s_1+f(1)+f(r+1)+f(2r+1)+f(3r+1) \\ &\vdots \end{aligned}$$

so, for $n=mr+1,\ m=1,2,3,...$

$$\begin{aligned} s_n=s_{mr+1} &= s_1+f(1)+f(r+1)+f(2r+1)+\cdots+f\big((m-1)r+1\big) \\ &= s_1+\sum_{k=0}^{m-1}f(kr+1) \end{aligned}$$

but

$$\begin{aligned} f(rk+1) &= (-1)^{rk+1}\ln\left|c_1+\sum_{j=1}^{\frac{r-1}{2}}\big[c_{2j}\cos\frac{2j(rk+1)\pi}{r}+c_{2j+1}\sin\frac{2j(rk+1)\pi}{r}\big]+\frac{b}{r}(rk+1)\right| \\ &= (-1)^{rk+1}\ln\left|c_1+\sum_{j=1}^{\frac{r-1}{2}}\big[c_{2j}\cos\frac{2j\pi}{r}+c_{2j+1}\sin\frac{2j\pi}{r}\big]+bk+\frac{b}{r}\right| \end{aligned}$$

hence,

$$s_{mr+1} = s_1 + \sum_{k=0}^{m-1} (-1)^{rk+1} \ln \left| c_1 + \sum_{j=1}^{\frac{r-1}{2}} \left[c_{2j} \cos \frac{2j\pi}{r} + c_{2j+1} \sin \frac{2j\pi}{r} \right] + bk + \frac{b}{r} \right|. \quad (3.17)$$

Also

$$\begin{aligned}
s_{r+2} &= s_2 + f(2) \\
s_{2r+2} &= s_{r+2} + f(r+2) = s_2 + f(2) + f(r+2) \\
s_{3r+2} &= s_{2r+2} + f(2r+2) = s_2 + f(2) + f(r+2) + f(2r+2) \\
s_{4r+2} &= s_{3r+2} + f(3r+2) = s_2 + f(2) + f(r+2) + f(2r+2) + f(3r+2) \\
&\vdots
\end{aligned}$$

so, for $n = mr + 2, \;\; m = 1, 2, 3, ...$

$$\begin{aligned}
s_n = s_{mr+2} &= s_2 + f(2) + f(r+2) + f(2r+2) + \cdots + f\big((m-1)r + 2\big) \\
&= s_2 + \sum_{k=0}^{m-1} f(kr+2)
\end{aligned}$$

but

$$\begin{aligned}
f(kr+2) &= (-1)^{kr+2} \ln \left| c_1 + \sum_{j=1}^{\frac{r-1}{2}} \left[c_{2j} \cos \frac{2j(kr+2)\pi}{r} + c_{2j+1} \sin \frac{2j(kr+2)\pi}{r} \right] + \frac{b}{r}(kr+2) \right| \\
&= (-1)^{kr} \ln \left| c_1 + \sum_{j=1}^{\frac{r-1}{2}} \left[c_{2j} \cos \frac{2j\pi}{r} + c_{2j+1} \sin \frac{2j\pi}{r} \right] + bk + \frac{2b}{r} \right|
\end{aligned}$$

hence,

$$s_{mr+2} = s_2 + (-1)^{kr} \ln \left| c_1 + \sum_{j=1}^{\frac{r-1}{2}} \left[c_{2j} \cos \frac{2j\pi}{r} + c_{2j+1} \sin \frac{2j\pi}{r} \right] + bk + \frac{2b}{r} \right|. \quad (3.18)$$

Recursively up to $n = mr + (r-1), \;\; m = 1, 2, \ldots,$ we get

$$s_n = s_{mr+(r-1)} = s_2 + \sum_{k=0}^{m-1} f\big(kr + (r-1)\big)$$

and

$$f\big(kr+(r-1)\big) = (-1)^{kr+(r-1)} \ln\left|c_1 + \sum_{j=1}^{\frac{r-1}{2}} \left[c_{2j}\cos\frac{2(r-1)j\pi}{r} + c_{2j+1}\sin\frac{2(r-1)j\pi}{r}\right] + bk + \frac{(r-1)b}{r}\right|$$

hence,

$$s_{mr+(r-1)} = (-1)^{kr+(r-1)} \ln\left|c_1 + \sum_{j=1}^{\frac{r-1}{2}} \left[c_{2j}\cos\frac{2(r-1)j\pi}{r} + c_{2j+1}\sin\frac{2(r-1)j\pi}{r}\right] + bk + \frac{(r-1)b}{r}\right| \quad (3.19)$$

Now, from (3.16,3.17,3.18,3.19) we obtain, for $n = mr+l$, $m = 1,2,\ldots$ and $l = 0,1,2,\ldots,r-1$

$$s_n = s_{mr+l} = s_l + \sum_{k=0}^{m-1}(-1)^{kr+l}\ln\left|c_1 + \sum_{j=1}^{\frac{r-1}{2}}\left[c_{2j}\cos\frac{2lj\pi}{r} + c_{2j+1}\sin\frac{2lj\pi}{r}\right] + bk + \frac{lb}{r}\right|.$$

The canonical coordinate

$$s_{mr+l} = (-1)^{mr+l}\ln|u_{mr+l}|, \qquad m = 1,2,3,\ldots \text{ and } l = 1,2,\ldots,(r-1)$$

which implies

$$\begin{aligned} u_{mr+l} &= \exp\big((-1)^{mr+l}s_{mr+l}\big) \\ &= u_l^{(-1)^{mr}}\prod_{k=0}^{m-1}\left(c_1 + \sum_{j=1}^{\frac{r-1}{2}}\left[c_{2j}\cos\frac{2lj\pi}{r} + c_{2j+1}\sin\frac{2lj\pi}{r}\right] + bk + \frac{lb}{r}\right)^{(-1)^{r(k+m)}} \end{aligned} \quad (3.20)$$

Now, to determine the forbidden set, from(3.20), $n = mr+l, l = 0,1,\ldots,r-1$

$$u_n = u_{mr+l} = u_l^{(-1)^{mr}}\prod_{k=0}^{m-1}\left(c_1 + \sum_{j=1}^{\frac{r-1}{2}}\left[c_{2j}\cos\frac{2lj\pi}{r} + c_{2j+1}\sin\frac{2lj\pi}{r}\right] + bk + \frac{lb}{r}\right)^{(-1)^{r(k+m)}}$$

let

$$f_l(m,k) = \left(c_1 + \sum_{j=1}^{\frac{r-1}{2}} \left[c_{2j}\cos\frac{2lj\pi}{r} + c_{2j+1}\sin\frac{2lj\pi}{r}\right] + bk + \frac{lb}{r}\right)^{(-1)^{r(k+m)}}$$

then

$$\begin{aligned} f_l(m,k) \quad = \quad & \Bigg(\left[\frac{1}{\Delta}\sum_{t=0}^{r-1}(-1)^t\Big(\frac{1}{u_tu_{t+r}} - \frac{tb}{r}\Big)\Delta_{t1}\right] \\ & + \sum_{j=1}^{\frac{r-1}{2}}\left[\frac{1}{\Delta}\sum_{t=0}^{r-1}(-1)^{t+1}\Big(\frac{1}{u_tu_{t+r}} - \frac{tb}{r}\Big)\Delta_{t(2j)}\right]\cos\frac{2jl\pi}{r} \\ & + \sum_{j=1}^{\frac{r-1}{2}}\left[\frac{1}{\Delta}\sum_{t=0}^{r-1}(-1)^t\Big(\frac{1}{u_tu_{t+r}} - \frac{tb}{r}\Big)\Delta_{t(2j+1)}\right]\sin\frac{2jl\pi}{r} + \frac{kbr+lb}{r}\Bigg)^{(-1)^{3(k+m)}} \end{aligned}$$

$$\begin{aligned} = \quad & \Bigg(\sum_{t=0}^{r-1}\frac{1}{u_tu_{t+r}}\Bigg(\frac{1}{\Delta}(-1)^t\Delta_{t1} + \sum_{j=1}^{\frac{r-1}{2}}\frac{1}{\Delta}(-1)^{t+1}\Delta_{t(2j)}\cos\frac{2jl\pi}{r} \\ & \qquad + \sum_{j=1}^{\frac{r-1}{2}}\frac{1}{\Delta}(-1)^t\Delta_{t(2j+1)}\sin\frac{2jl\pi}{r}\Bigg) \\ & -\frac{1}{r}\sum_{t=0}^{r-1}tb\Bigg(\frac{1}{\Delta}(-1)^t\Delta_{t1} + \sum_{j=1}^{\frac{r-1}{2}}\frac{1}{\Delta}(-1)^{t+1}\Delta_{t(2j)}\cos\frac{2jl\pi}{r} \\ & \qquad + \sum_{j=1}^{\frac{r-1}{2}}\frac{1}{\Delta}(-1)^t\Delta_{t(2j+1)}\ \sin\frac{2jl\pi}{r} - kbr - lb\Bigg)\Bigg)^{(-1)^{3(k+m)}} \end{aligned} \tag{3.21}$$

$$= \left(\sum_{t=0}^{r-1} \frac{u_0u_1u_2\dots u_{2r-1}r/u_tu_{t+r}}{u_0u_1u_2\dots u_{2r-1}r} \left(\frac{1}{\Delta}(-1)^t\Delta_{t1} + \sum_{j=1}^{\frac{r-1}{2}} \frac{1}{\Delta}(-1)^{t+1}\Delta_{t(2j)} \cos\frac{2jl\pi}{r} + \sum_{j=1}^{\frac{r-1}{2}} \frac{1}{\Delta}(-1)^t\Delta_{t(2j+1)} \ \sin\frac{2jl\pi}{r} \right) - \frac{u_0u_1u_2\dots u_{2r-1}}{u_0u_1u_2\dots u_{2r-1}r} \sum_{t=0}^{r-1} \left(\frac{1}{\Delta}(-1)^t\Delta_{t1} + \sum_{j=1}^{\frac{r-1}{2}} \frac{1}{\Delta}(-1)^{t+1}\Delta_{t(2j)} \cos\frac{2jl\pi}{r} + \sum_{j=0}^{\frac{r-1}{2}} \frac{1}{\Delta}(-1)^t\Delta_{t(2j+1)} \ \sin\frac{2jl\pi}{r} - kbr - lb \right) \right)^{(-1)^{3(k+m)}} \tag{3.22}$$

such that $u_0u_1u_2\dots u_{2r-1}r \neq 0$. We have the following results,

1. If $k+m =$ even number for some m and k, from (3.22) we have $f_l(m,k)$ is defined where
$$u_0u_1u_2\dots u_{2r-1} \neq 0$$

2. If $k+m =$ odd number for some m and k, from (3.21) we have $f_l(m,k)$ is undefined when

$$
\begin{aligned}
0 &= \sum_{t=0}^{r-1} \frac{1}{u_t u_{t+r}} \Bigg(\frac{1}{\Delta}(-1)^t \Delta_{t1} + \sum_{j=1}^{\frac{r-1}{2}} \frac{1}{\Delta}(-1)^{t+1} \Delta_{t(2j)} \cos \frac{2jl\pi}{r} \\
&\qquad + \sum_{j=0}^{\frac{r-1}{2}} \frac{1}{\Delta}(-1)^t \Delta_{t(2j+1)} \sin \frac{2jl\pi}{r} \Bigg) \\
&\quad - \frac{1}{r} \sum_{t=0}^{r-1} tb \Bigg(\frac{1}{\Delta}(-1)^t \Delta_{t1} + \sum_{j=1}^{\frac{r-1}{2}} \frac{1}{\Delta}(-1)^{t+1} \Delta_{t(2j)} \cos \frac{2jl\pi}{r} \\
&\qquad + \sum_{j=0}^{\frac{r-1}{2}} \frac{1}{\Delta}(-1)^t \Delta_{t(2j+1)} \sin \frac{2jl\pi}{r} - kbr - lb \Bigg) \\
&= \sum_{t=0}^{r-1} \frac{1}{u_t u_{t+r}} \xi_{tk}^l - \frac{1}{r} \xi_k^l
\end{aligned}
$$

where

$$
\begin{aligned}
\xi_{tk}^l &= \Bigg(\frac{1}{\Delta}(-1)^t \Delta_{t1} + \sum_{j=1}^{\frac{r-1}{2}} \frac{1}{\Delta}(-1)^{t+2j+1} \Delta_{t(2j)} \cos \frac{2jl\pi}{r} \\
&\qquad + \sum_{j=0}^{\frac{r-1}{2}} \frac{1}{\Delta}(-1)^{t+2j} \Delta_{t(2j+1)} \sin \frac{2jl\pi}{r} \Bigg) \\
\xi_k^l &= \sum_{t=0}^{r-1} tb \Bigg(\frac{1}{\Delta}(-1)^t \Delta_{t1} + \sum_{j=1}^{\frac{r-1}{2}} \frac{1}{\Delta}(-1)^{t+2j+1} \Delta_{t(2j)} \cos \frac{2jl\pi}{r} \\
&\qquad + \sum_{j=0}^{\frac{r-1}{2}} \frac{1}{\Delta}(-1)^{t+2j} \Delta_{t(2j+1)} \sin \frac{2jl\pi}{r} - kbr - lb \Bigg)
\end{aligned}
$$

Theorem 3.1.2. *Let $u_0, u_1, u_2, \ldots, u_{2r-1}$ be given real numbers such that $u_0 u_1 u_2 \ldots u_{2r-1} \neq 0$ and let $a \neq 1$. Then the forbidden set $\mathcal{F}$ of the difference equation (3.1)is given by $F = \mathcal{F}^0 \bigcup \mathcal{F}^1 \bigcup \cdots \bigcup \mathcal{F}^{r-1}$ where*

$$
\mathcal{F}^l = \bigcup_{k=1}^{\infty} \left\{ \sum_{t=0}^{r-1} \left(\frac{1}{u_t u_{t+r}} \xi_{tk} - \frac{1}{r} \xi_k \right) = 0 \right\}
$$

3.2 Dynamics

In this section we study the dynamics of difference equation (3.1), we concentrate on the equilibrium points and asymptotic stability of this equilibrium points.

3.2.1 Equilibrium points and stability

We investigate the equilibrium points of the difference equation (3.1) where a, b are real numbers such that $a < 1$ and the initial condition $u_0, u_1, \ldots, u_{n+2i+1}$ are also real numbers.

Definition 9. *[7]* The equilibrium point $\bar{u}$ of the difference equation

$$u_{n+k} = f(u_0, u_1, \ldots, u_{n+k-1}), \qquad n = 0, 1, \ldots \tag{3.23}$$

is the point that satisfies the condition

$$\bar{u} = f(\bar{u}, \bar{u}, \ldots, \bar{u}).$$

The equilibrium point of our difference equation (3.1) is

$$\bar{u} = \frac{\bar{u}}{a + b\bar{u}^2}$$

then,

$$\bar{u} = 0 \qquad \text{or} \qquad \bar{u} = \pm\sqrt{\frac{1-a}{b}},$$

Definition 10. *[7]* Let $\bar{u}$ be an equilibrium point of equation (3.23)

(a) The equilibrium $\bar{u}$ is called locally stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\{u_n\}_{n=0}^{\infty}$ is a solution of equation (3.23) with

$$|u_0 - \bar{u}| + \cdots + |u_{n+k-1} - \bar{u}| < \delta$$

then

$$|u_n - \bar{u}| < \varepsilon, \quad \text{for all} \quad n \geq 0.$$

(b) The equilibrium $\bar{u}$ is called locally asymptotically stable if it is locally stable and if there exists $\gamma > 0$ such that if $\{u_n\}_{n=0}^{\infty}$ is a solution of equation (3.23) with

$$|u_0 - \bar{u}| + \ldots + |u_{n+k-1} - \bar{u}| < \gamma$$

then

$$\lim_{n\to\infty} u_n = \bar{u}$$

(c) The equilibrium $\bar{u}$ is called a global attractor if for every solution $\{u_n\}_{n=0}^{\infty}$ equation (3.23), we have

$$\lim_{n\to\infty} u_n = \bar{u}$$

(d) The equilibrium $\bar{u}$ is called global asymptotically stable if it is locally stable and is a global attractor .

(e) The equilibrium $\bar{u}$ is unstable if it is not stable.

(f) The equilibrium point $\bar{u}$ is a source or a repeller, if there exists $r > 0$ such that for every solution $\{u_n\}_{n=0}^{\infty}$ equation (3.23), with

$$|u_0 - \bar{u}| + |u_1 - \bar{u}| + \cdots + |u_{n+k-1} - \bar{u}| < r$$

there exists $N \geq 1$ such that

$$|u_N - \bar{u}| \geq r$$

Clearly, a repeller is an unstable equilibrium.

3.2.2 Local Stability of The Equilibrium Points

To study the stability of the equilibrium points, we find the linearized equation of the difference equation.

Definition 11. *[7]* The linearized equation of difference equation $u_{n+k} = f(u_{n+k-1}, \ldots, u_n)$, of order k, about the equilibrium point $\bar{x}$ is defined by the equation

$$u_{n+k} = \rho_0 z_{n+k-1} + \rho_1 z_{n+k-2} + \cdots + \rho_{k-1} z_n,$$

where

$$\rho_j = \frac{\partial f(\bar{u}, \bar{u}, \ldots, \bar{u})}{\partial u_{n-j}}, \quad j = 0, 1, \ldots, k-1$$

Theorem 3.2.1. *[7]* Assume $q_j \in \mathbb{R}, j = 0, a, \ldots k-1$, then

$$\sum_{j=0}^{k-1} |q_j| < 1$$

is a sufficient condition for the asymptotic stability of the difference equation

$$z_{n+k} + q_1 z_{n+k-1} + \cdots + q_k z_n = 0, \qquad n = 0, 1, 2, \ldots$$

From previous definition, we find the linearization of our difference equation (3.1). Consider

$$w(x, y) = \frac{x}{a + byx},$$

then

$$w_x(x, y) = \frac{a + byx - byx}{(a + byx)^2} = \frac{a}{(a + byx)^2},$$

and

$$w_y(x, y) = \frac{-byx}{(a + byx)^2}$$

which implies

$$w_x(\bar{u}, \bar{u}) = \frac{a}{(a + b\bar{u}^2)^2}$$

and

$$w_y(\bar{u}, \bar{u}) = \frac{-b\bar{u}^2}{(a + b\bar{u}^2)^2}$$

The linearized equation is

$$z_{n+2(i+1)} = w_y(\bar{u}, \bar{u})z_{n+i+1} + w_x(\bar{u}, \bar{u})z_n$$

$$z_{n+2(i+1)} = \frac{-b\bar{u}^2}{(a + b\bar{u}^2)^2}z_{n+i+1} + \frac{a}{(a + b\bar{u}^2)^2}z_n$$

i.e.

$$z_{n+2(i+1)} + \frac{b\bar{u}^2}{(a + b\bar{u}^2)^2}z_{n+i+1} - \frac{a}{(a + b\bar{u}^2)^2}z_n = 0 \tag{3.24}$$

Local Stability of The Zero Equilibrium Point

By substituting $\bar{u} = 0$ in the linearized equation (3.24), we get

$$z_{n+2(i+1)} - \frac{1}{a}z_n = 0.$$

By applying theorem (3.2.1) to the equation, we get

$$\left|\frac{1}{a}\right| < 1$$

is a sufficient for the asymptotically stable of the difference equation, then

$$a \in (-\infty, -1) \cap (1, \infty)$$

but $a < 1$, thus

$$-\infty < a < -1$$

Theorem 3.2.2. *Assume that* $-\infty < a < -1$*, then the zero equilibrium point of equation* (3.1) *is local asymptotically stable.*

Local stability of the equilibrium point $\bar{u} = +\sqrt{\frac{1-a}{b}}$

By substituting $\bar{u} = \sqrt{\frac{1-a}{b}}$ in the linearized equation (3.24),we get

$$z_{n+2(i+1)} + (1-a)z_{n+i+1} - az_n = 0,$$

apply theorem (3.2.1) to the equation, we get

$$|1-a| + |a| < 1$$

is a sufficient for the asymptotically stable of the difference equation. This sufficient condition never hold for all a. By the same way, we get not satisfying sufficient condition for the reminder negative equilibrium point.

3.3 The case $i = 0$

$$u_{n+2} = \frac{u_n}{a + bu_{n+1}u_n} = w(u_n, u_{n+1}), \quad u_0u_1 \neq 0. \tag{3.25}$$

From previous chapter in example (11), the solution of equation (3.25) is

Case 1 : if $a \neq 1$

$$u_n = u_0^{(-1)^n} \prod_{k=0}^{n-1} \left(a^k \frac{1}{u_0u_1} + b\left[\frac{a^k - 1}{a-1}\right]\right)^{(-1)^{k+n}}. \tag{3.26}$$

Now, to determine the forbidden set. From(3.26)

$$\begin{aligned} u_n &= u_0^{(-1)^n} \cdot \prod_{k=0}^{n-1} \left(a^k \frac{1}{u_0u_1} + b\left[\frac{a^k - 1}{a-1}\right]\right)^{(-1)^{k+n}} \\ &= u_0^{(-1)^n} \cdot \prod_{k=0}^{n-1} \left(\frac{a^k(a-1) + bu_0u_1(a^k-1)}{u_0u_1(a-1)}\right)^{(-1)^{k+n}} \end{aligned}$$

let

$$f(n,k) = \left(\frac{a^k(a-1) + bu_0u_1(a^k-1)}{u_0u_1(a-1)}\right)^{(-1)^{k+n}}$$

so

$$u_n = u_0^{(-1)^n} \cdot \prod_{k=0}^{n-1} f(n,k), \qquad n = 2,3,4,...$$

1. if $k+n =$ even number then $f(n,k)$ is defined when

$$u_0u_1 \neq 0$$

2. if $k+n =$ odd number them so $f(n,k)$ is undefined when

$$a^k(a-1) + bu_0u_1(a^k-1) = 0$$

so

$$u_0u_1 = -\frac{a^k(a-1)}{b(a^k-1)}.$$

Theorem 3.3.1. *Let u_0, u_1 be given initial conditions such that $u_0u_1 \neq 0$ and let $a \neq 1$. The forbidden set $\mathcal{F}$ of the difference equation* (3.25) *is given by*

$$\mathcal{F} = \bigcup_{k=1}^{\infty} \left\{ u_0u_1 = -\frac{a^k(a-1)}{b(a^k-1)} \right\}$$

Case 2 : if $a = 1$

$$u_n = u_0^{(-1)^n} \prod_{k=0}^{n-1} \big(\frac{1}{u_0u_1} + bk\big)^{(-1)^{k+n}}.$$

To find the forbidden set. From (3.27)

$$\begin{aligned} u_n &= u_0^{(-1)^n} \prod_{k=0}^{n-1} \big(\frac{1}{u_0u_1} + bk\big)^{(-1)^{k+n}} \\ &= u_0^{(-1)^n} \prod_{k=0}^{n-1} \left(\frac{1+u_0u_1bk}{u_0u_1}\right)^{(-1)^{k+n}} \end{aligned}$$

let

$$f(n,k) = \left(\frac{1+u_0u_1bk}{u_0u_1}\right)^{(-1)^{k+n}}$$

so

$$u_n = u_0^{(-1)^n} \prod_{k=0}^{n-1} f(n,k), \qquad n = 2,3,4,...$$

1. if $k+n =$ even number then $f(n,k)$ is defined where
$$u_0 u_1 \neq 0.$$

2. if $k+n =$ odd number then $f(n,k)$ is undefined when
$$1 + u_0 u_1 bk = 0$$
so
$$u_0 u_1 = -\frac{1}{bk}.$$

Theorem 3.3.2. *Let u_0, u_1 be given such that $u_0 u_1 \neq 0$ and let $a = 1$. The forbidden set $\mathcal{F}$ of the difference equation* (3.25)*is given by*

$$\mathcal{F} = \bigcup_{k=1}^{\infty} \left\{ u_0 u_1 = -\frac{1}{bk} \right\}$$

3.4 Special Case for $i = 2$

In this section, we find the exact solution and the forbidden set of the following 6^{th} order difference equation

$$u_{n+6} = \frac{u_n}{a + bu_{n+3}u_n} = w, \quad u_0 u_1 u_2 u_3 u_4 u_5 \neq 0 \tag{3.27}$$

Let us differentiate w with respect to u_n and u_{n+3}

$$\begin{aligned} \frac{\partial w}{\partial u_n} &= \frac{a + bu_n u_{n+3} - bu_n u_{n+3}}{(a + bu_n u_{n+3})^2} = \frac{a}{(a + bu_n u_{n+3})^2} . \frac{u_n^2}{u_n^2} = \frac{aw^2}{u_n^2}, \\ \frac{\partial w}{\partial u_{n+3}} &= \frac{-bu_n^2}{(a + bu_n u_{n+3})^2} = -bw^2 \end{aligned}$$

and so

$$\frac{\partial u_{n+3}}{\partial u_n} = -\frac{\partial w / \partial u_n}{\partial w / \partial u_{n+3}} = -\frac{aw^2/u_n^2}{-bw^2} = \frac{a}{bu_n^2}.$$

The linearized symmetry condition (LSC) is given by

$$Q(n+6, u_{n+6}) - \frac{\partial w}{\partial u_{n+3}} Q(n+3, u_{n+3}) - \frac{\partial w}{\partial u_n} Q(n, u_n) = 0$$

$$Q(n+6, u_{n+6}) + bw^2 Q(n+3, u_{n+3}) - \frac{aw^2}{u_n^2} Q(n, u_n) = 0.$$

Now, by applying the differential operator L to the previous equation, where

$$L = \frac{\partial}{\partial u_n} + \frac{\partial u_{n+3}}{\partial u_n} \frac{\partial}{\partial u_{n+3}}$$

we get

$$\frac{\partial}{\partial u_n}\Big(Q(n+6, u_{n+6})\Big) + \frac{\partial u_{n+3}}{\partial u_n} \frac{\partial}{\partial u_{n+3}}\Big(Q(n+6, u_{n+6})\Big)$$

$$= \frac{\partial}{\partial u_n}\Big(- bw^2 Q(n+3, u_{n+3}) + \frac{aw^2}{u_n^2} Q(n, u_n)\Big) + \frac{a}{bu_n^2} \frac{\partial}{\partial u_{n+3}}\Big(- bw^2 Q(n+3, u_{n+3}) + \frac{aw^2}{u_n^2} Q(n, u_n)\Big)$$

which implies

$$\frac{aw^2}{u_n^2} Q'(n, u_n) - \frac{2aw^2}{u_n^3} Q(n, u_n) - \frac{aw^2}{u_n^2} Q'(n+3, u_{n+3}) = 0$$

multiply this equation by $\frac{u_n^2}{aw^2}$

$$Q'(n, u_n) - \frac{2}{u_n} Q(n, u_n) - Q'(n+3, u_{n+3}) = 0 \tag{3.28}$$

differentiate the last equation with respect to u_n keeping u_{n+3} fixed

$$Q''(n, u_n) - \frac{2}{u_n} Q'(n, u_n) + \frac{2}{u_n^2} Q(n, u_n) = 0$$

again multiply by u_n^2

$$u_n^2 Q''(n, u_n) - 2u_n Q'(n, u_n) + 2Q(n, u_n) = 0$$

which is an Euler Equation, whose solution is

$$Q(n, u_n) = \alpha_n u_n^2 + \beta_n u_n$$

thus

$$Q'(n,u_n) = 2\alpha_n u_n + \beta_n$$

substitute into equation (3.28)

$$\begin{aligned} 0 &= 2\alpha_n u_n + \beta_n - 2\alpha_n u_n - 2\beta_n - 2\alpha_{n+3}u_{n+3} - \beta_{n+3} \\ &= -\beta_n - \beta_{n+3} - 2\alpha_{n+3}u_{n+3} \end{aligned}$$

comparing both sides of the last equation, we get

$$\alpha_{n+3} = 0 \text{ and so } \ \alpha_n = 0$$

and we have also

$$\beta_{n+3} + \beta_n = 0$$

which is a third order linear difference equation, whose solution is

$$\beta_n = c_1(-1)^n + c_2\left(\frac{1+\sqrt{3}i}{2}\right)^n + c_3\left(\frac{1-\sqrt{3}i}{2}\right)^n$$

where c_1, c_2, c_3 are constants. Suppose that $c_2 = c_3 = 0$ and $c_1 = 1$ so $\beta_n = (-1)^n$, which implies

$$Q(n,u_n) = (-1)^n u_n.$$

We want to find the invariant using,

$$\frac{du_n}{(-1)^n u_n} = \frac{du_{n+3}}{(-1)^{n+3}u_{n+3}} := \frac{dv_n}{0},$$

take $\frac{du_n}{(-1)^n u_n} = \frac{du_{n+3}}{(-1)^{n+3}u_{n+3}}$ invariants,
so

$$\ln \mid u_n \mid = -\ln \mid u_{n+3} \mid + c, \quad \text{then} \quad c = \ln \mid u_n u_{n+3} \mid$$

where c is constant, so

$$k_1 = u_n u_{n+3} \text{ where } k_1 = e^c,$$

we also have

$$\frac{du_n}{u_n} := \frac{dv_n}{0}$$

and so

$$v_n = k, \quad \text{such that } k = f(k_1)$$

where k, k_1 are constants. Let $f(k_1) = k_1$, then

$$v_n = u_{n+3}u_n,$$

and

$$v_{n+3} = u_{n+6}u_{n+3} = \frac{u_n u_{n+3}}{a + bu_n u_{n+3}} = \frac{v_n}{a + bv_n}.$$

Now, we want to solve this equation

$$v_{n+3} = \frac{v_n}{a + bv_n} \tag{3.29}$$

so

$$\frac{1}{v_{n+3}} = \frac{a}{v_n} + b, \tag{3.30}$$

let

$$z_n = \frac{1}{v_n},$$

this substitution converts equation (3.27) to the following third order linear equation

$$z_{n+3} - az_n - b = 0 \tag{3.31}$$

so

$$z_{n+3} - az_n = b. \tag{3.32}$$

3.4.1 The Case $a \neq 1$

The characteristic equation of the homogeneous equation

$$z_{n+3} - az_n = 0 \tag{3.33}$$

is

$$\lambda^3 - a = 0.$$

The roots of the last equation are

$$\begin{aligned}
\mu_1 &= (a)^{\frac{1}{3}} e^{i\frac{2(0)\pi}{3}} = (a)^{\frac{1}{3}} \text{ ,real root,} \\
\mu_2 &= (a)^{\frac{1}{3}} e^{i\frac{2(1)\pi}{3}} = (a)^{\frac{1}{3}} e^{i\frac{2\pi}{3}} \\
\mu_3 &= (a)^{\frac{1}{3}} e^{i\frac{2(2)\pi}{3}} = (a)^{\frac{1}{3}} e^{i\frac{4\pi}{3}}
\end{aligned}$$

where $(a)^{\frac{1}{3}}$ is the real positive third root of a. So the solution of the homogeneous equation (3.33) is

$$\begin{aligned} z_h &= c_1\mu_1^n + \acute{c_2}\mu_2^n + \acute{c_3}\mu_3^n \\ &= c_1(a)^{\frac{n}{3}} + \acute{c_2}\Big((a)^{\frac{1}{3}}e^{i\frac{2\pi}{3}}\Big)^n + \acute{c_3}\Big((a)^{\frac{1}{3}}e^{i\frac{4\pi}{3}}\Big)^n \\ &= c_1(a)^{\frac{n}{3}} + \acute{c_2}\big((a)^{\frac{n}{3}}(\cos\frac{2n\pi}{3} + i\sin\frac{2n\pi}{3})\big) + \acute{c_3}\big((a)^{\frac{1}{3}}(\cos\frac{2n\pi}{3} - i\sin\frac{2n\pi}{3})\big) \\ &= a^{\frac{n}{3}}\big(c_1 + c_2\cos\frac{2n\pi}{3} + c_3\sin\frac{2n\pi}{3}\big) \end{aligned}$$

where $c_1,\ \acute{c_2},\ \acute{c_3}$ are constants and

$$c_2 = \acute{c_2} + \acute{c_3},\ c_3 = i(\acute{c_2} - \acute{c_3})$$

to find the particular solution,z_p, let

$$z_p = c,$$

substitute into equation (3.32), we obtain

$$c - ac = b$$

so

$$c = \frac{b}{1-a}$$

such that $b \neq 0$, and so

$$z_p = \frac{b}{1-a}$$

thus, the solution of equation (3.31) is

$$\begin{aligned} z_n &= z_h + z_p \\ &= a^{\frac{n}{3}}\big(c_1 + c_2\cos\frac{2n\pi}{3} + c_3\sin\frac{2n\pi}{3}\big) + \frac{b}{1-a} \end{aligned}$$

Given initial values z_0, z_1, z_2 , the constants c_1, c_2, c_3 satisfy the following system of equations

$$\begin{aligned} \big(z_0 - \frac{b}{1-a}\big) &= c_1 + c_2 \\ \big(z_0 - \frac{b}{1-a}\big)a^{-\frac{1}{3}} &= c_1 + c_2\cos\frac{2\pi}{3} + c_3\sin\frac{2\pi}{3} \\ \big(z_0 - \frac{b}{1-a}\big)a^{-\frac{2}{3}} &= c_1 + c_2\cos\frac{4\pi}{3} + c_3\sin\frac{4\pi}{3} \end{aligned}$$

then

$$c_1 = \frac{\Delta_1}{\Delta}, \quad c_2 = \frac{\Delta_2}{\Delta} \text{ and } c_3 = \frac{\Delta_3}{\Delta}$$

such that

$$\Delta = \begin{vmatrix} 1 & 1 & 0 \\ 1 & \cos\frac{2\pi}{3} & \sin\frac{2\pi}{3} \\ 1 & \cos\frac{4\pi}{3} & \sin\frac{4\pi}{3} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ 1 & \frac{-1}{2} & \frac{-\sqrt{3}}{2} \end{vmatrix} = \frac{3\sqrt{3}}{2}$$

$$\Delta_1 = \begin{vmatrix} \left(z_0 - \frac{b}{1-a}\right) & 1 & 0 \\ \left(z_1 - \frac{b}{1-a}\right)a^{-\frac{1}{3}} & \cos\frac{2\pi}{3} & \sin\frac{2\pi}{3} \\ \left(z_2 - \frac{b}{1-a}\right)a^{-\frac{2}{3}} & \cos\frac{4\pi}{3} & \sin\frac{4\pi}{3} \end{vmatrix} = \begin{vmatrix} \left(z_0 - \frac{b}{1-a}\right) & 1 & 0 \\ \left(z_1 - \frac{b}{1-a}\right)a^{-\frac{1}{3}} & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \left(z_2 - \frac{b}{1-a}\right)a^{-\frac{2}{3}} & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{vmatrix}^T$$

$$\begin{aligned}
&= \begin{vmatrix} \left(z_0 - \frac{b}{1-a}\right) & \left(z_1 - \frac{b}{1-a}\right)a^{-\frac{1}{3}} & \left(z_2 - \frac{b}{1-a}\right)a^{-\frac{2}{3}} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{vmatrix} \\
&= z_0\frac{\sqrt{3}}{2} + z_1\left(-\frac{\sqrt{3}}{2}a^{-\frac{1}{3}}\right) + z_2\left(\frac{\sqrt{3}}{2}a^{-\frac{2}{3}}\right) - \frac{b}{1-a}\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}a^{-\frac{1}{3}} + \frac{\sqrt{3}}{2}a^{-\frac{2}{3}}\right) \\
&= \frac{1}{u_0u_3}\frac{\sqrt{3}}{2} + \frac{1}{u_1u_4}\left(-\frac{\sqrt{3}}{2}a^{-\frac{1}{3}}\right) + \frac{1}{u_2u_5}\left(\frac{\sqrt{3}}{2}a^{-\frac{2}{3}}\right) - \frac{b}{1-a}\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}a^{-\frac{1}{3}} + \frac{\sqrt{3}}{2}a^{-\frac{2}{3}}\right)
\end{aligned}$$

so

$$\begin{aligned}
c_1 &= \frac{\Delta_1}{\Delta} \\
&= \frac{\frac{1}{u_0u_3}\frac{\sqrt{3}}{2} + \frac{1}{u_1u_4}\left(-\frac{\sqrt{3}}{2}a^{-\frac{1}{3}}\right) + \frac{1}{u_2u_5}\left(\frac{\sqrt{3}}{2}a^{-\frac{2}{3}}\right) - \frac{b}{1-a}\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}a^{-\frac{1}{3}} + \frac{\sqrt{3}}{2}a^{-\frac{2}{3}}\right)}{3\sqrt{3}/2} \\
&= \frac{1}{u_0u_3}\frac{1}{3} + \frac{1}{u_1u_4}\left(-\frac{1}{3}a^{-\frac{1}{3}}\right) + \frac{1}{u_2u_5}\left(\frac{1}{3}a^{-\frac{2}{3}}\right) - \frac{b}{1-a}\left(\frac{1}{3} - \frac{1}{3}a^{-\frac{1}{3}} + \frac{1}{3}a^{-\frac{2}{3}}\right) \\
&= \frac{1}{3}\left[\frac{1}{u_0u_3} + \frac{1}{u_1u_4}\left(-a^{-\frac{1}{3}}\right) + \frac{1}{u_2u_5}\left(a^{-\frac{2}{3}}\right) - \frac{b}{1-a}\left(1 - a^{-\frac{1}{3}} + a^{-\frac{2}{3}}\right)\right]
\end{aligned}$$

$$\begin{aligned}
\Delta_2 &= \begin{vmatrix} 1 & \left(z_0 - \frac{b}{1-a}\right) & 0 \\ 1 & \left(z_1 - \frac{b}{1-a}\right)a^{-\frac{1}{3}} & \sin\frac{2\pi}{3} \\ 1 & \left(z_2 - \frac{b}{1-a}\right)a^{-\frac{2}{3}} & \sin\frac{4\pi}{3} \end{vmatrix} = \begin{vmatrix} 1 & \left(z_0 - \frac{b}{1-a}\right) & 0 \\ 1 & \left(z_1 - \frac{b}{1-a}\right)a^{-\frac{1}{3}} & \frac{\sqrt{3}}{2} \\ 1 & \left(z_2 - \frac{b}{1-a}\right)a^{-\frac{2}{3}} & -\frac{\sqrt{3}}{2} \end{vmatrix}^T \\
&= \begin{vmatrix} 1 & 1 & 1 \\ \left(z_0 - \frac{b}{1-a}\right) & \left(z_1 - \frac{b}{1-a}\right)a^{-\frac{1}{3}} & \left(z_2 - \frac{b}{1-a}\right)a^{-\frac{2}{3}} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{vmatrix} \\
&= -\begin{vmatrix} \left(z_0 - \frac{b}{1-a}\right) & \left(z_1 - \frac{b}{1-a}\right)a^{-\frac{1}{3}} & \left(z_2 - \frac{b}{1-a}\right)a^{-\frac{2}{3}} \\ 1 & 1 & 1 \\ 0 & \frac{\sqrt{3}}{2} & \frac{-\sqrt{3}}{2} \end{vmatrix} \\
&= z_0\sqrt{3} + z_1\left(-\frac{\sqrt{3}}{2}a^{-\frac{1}{3}}\right) + z_2\left(-\frac{\sqrt{3}}{2}a^{-\frac{2}{3}}\right) - \frac{b}{1-a}\left(\sqrt{3} - \frac{\sqrt{3}}{2}a^{-\frac{1}{3}} - \frac{\sqrt{3}}{2}a^{-\frac{2}{3}}\right) \\
&= \frac{1}{u_0u_3}\sqrt{3} + \frac{1}{u_1u_4}\left(-\frac{\sqrt{3}}{2}a^{-\frac{1}{3}}\right) + \frac{1}{u_2u_5}\left(-\frac{\sqrt{3}}{2}a^{-\frac{2}{3}}\right) - \frac{b}{1-a}\left(\sqrt{3} - \frac{\sqrt{3}}{2}a^{-\frac{1}{3}} - \frac{\sqrt{3}}{2}a^{-\frac{2}{3}}\right)
\end{aligned}$$

so

$$\begin{aligned}
c_2 &= \frac{\Delta_2}{\Delta} \\
&\quad \frac{\frac{1}{u_0u_3}\sqrt{3} + \frac{1}{u_1u_4}\left(-\frac{\sqrt{3}}{2}a^{-\frac{1}{3}}\right) + \frac{1}{u_2u_5}\left(-\frac{\sqrt{3}}{2}a^{-\frac{2}{3}}\right) - \frac{b}{1-a}\left(\sqrt{3} - \frac{\sqrt{3}}{2}a^{-\frac{1}{3}} - \frac{\sqrt{3}}{2}a^{-\frac{2}{3}}\right)}{3\sqrt{3}/2} \\
&= \frac{1}{3}\left[\frac{2}{u_0u_3} + \frac{1}{u_1u_4}\left(-a^{-\frac{1}{3}}\right) + \frac{1}{u_2u_5}\left(-a^{-\frac{2}{3}}\right) - \frac{b}{1-a}\left(2 - a^{-\frac{1}{3}} - a^{-\frac{2}{3}}\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
\Delta_3 &= \begin{vmatrix} 1 & 1 & \left(z_0 - \frac{b}{1-a}\right) \\ 1 & \cos\frac{2\pi}{3} & \left(z_1 - \frac{b}{1-a}\right)a^{-\frac{1}{3}} \\ 1 & \cos\frac{4\pi}{3} & \left(z_2 - \frac{b}{1-a}\right)a^{-\frac{2}{3}} \end{vmatrix} = \begin{vmatrix} 1 & 1 & \left(z_0 - \frac{b}{1-a}\right) \\ 1 & -\frac{1}{2} & \left(z_1 - \frac{b}{1-a}\right)a^{-\frac{1}{3}} \\ 1 & -\frac{1}{2} & \left(z_2 - \frac{b}{1-a}\right)a^{-\frac{2}{3}} \end{vmatrix}^T \\
&= \begin{vmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ \left(z_0 - \frac{b}{1-a}\right) & \left(z_1 - \frac{b}{1-a}\right)a^{-\frac{1}{3}} & \left(z_2 - \frac{b}{1-a}\right)a^{-\frac{2}{3}} \end{vmatrix} \\
&= -\begin{vmatrix} \left(z_0 - \frac{b}{1-a}\right) & \left(z_1 - \frac{b}{1-a}\right)a^{-\frac{1}{3}} & \left(z_2 - \frac{b}{1-a}\right)a^{-\frac{2}{3}} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & 1 & 1 \end{vmatrix} \\
&= z_1\left(\frac{3}{2}a^{-\frac{1}{3}}\right) + z_2\left(-\frac{3}{2}a^{-\frac{2}{3}}\right) - \frac{b}{1-a}\left(\frac{3}{2}a^{-\frac{1}{3}} - \frac{3}{2}a^{-\frac{2}{3}}\right) \\
&= \frac{1}{u_1u_4}\left(\frac{3}{2}a^{-\frac{1}{3}}\right) + \frac{1}{u_2u_5}\left(-\frac{3}{2}a^{-\frac{2}{3}}\right) - \frac{b}{1-a}\left(\frac{3}{2}a^{-\frac{1}{3}} - \frac{3}{2}a^{-\frac{2}{3}}\right)
\end{aligned}$$

so

$$\begin{aligned}
c_3 &= \frac{\Delta_3}{\Delta} \\
&= \frac{\frac{1}{u_1u_4}\left(\frac{3}{2}a^{-\frac{1}{3}}\right) + \frac{1}{u_2u_5}\left(-\frac{3}{2}a^{-\frac{2}{3}}\right) - \frac{b}{1-a}\left(\frac{3}{2}a^{-\frac{1}{3}} - \frac{3}{2}a^{-\frac{2}{3}}\right)}{3\sqrt{3}/2} \\
&= \frac{1}{\sqrt{3}}\left[\frac{1}{u_1u_4}\left(a^{-\frac{1}{3}}\right) + \frac{1}{u_2u_5}\left(-a^{-\frac{2}{3}}\right) - \frac{b}{1-a}\left(a^{-\frac{1}{3}} - a^{-\frac{2}{3}}\right)\right]
\end{aligned}$$

Then, we have

$$\begin{aligned}
v_n &= \frac{1}{z_n} \\
&= \frac{1}{a^{\frac{n}{3}}\left(c_1 + c_2\cos\frac{2n\pi}{3} + c_3\sin\frac{2n\pi}{3}\right) + \frac{b}{1-a}} = u_nu_{n+3}
\end{aligned}$$

and so

$$u_{n+3} = \frac{1}{u_n\left(a^{\frac{n}{3}}\left[c_1 + c_2\cos\frac{2n\pi}{3} + c_3\sin\frac{2n\pi}{3}\right] + \frac{b}{1-a}\right)}.$$

To solve the last equation we need to obtain the canonical coordinate,

$$s_n = \int \frac{du_n}{(-1)^n u_n}$$

$$= (-1)^n \ln|u_n|$$

so

$$\begin{aligned} s_{n+3} - s_n &= (-1)^{n+3}\ln|u_{n+3}| - (-1)^n \ln|u_n| \\ &= (-1)^{n+1}[\ln|u_{n+3}| + \ln|u_n|] \\ &= (-1)^{n+1}\ln|u_{n+3}u_n| \\ &= (-1)^{n+1}\ln|v_n| \\ &= (-1)^{n+1}\ln\left|\frac{1}{a^{\frac{n}{3}}\left(c_1 + c_2\cos\frac{2n\pi}{3} + c_3\sin\frac{2n\pi}{3}\right) + \frac{b}{1-a}}\right| \\ &= (-1)^{n+2}\ln\left|a^{\frac{n}{3}}\left(c_1 + c_2\cos\frac{2n\pi}{3} + c_3\sin\frac{2n\pi}{3}\right) + \frac{b}{1-a}\right| \end{aligned}$$

and so

$$s_{n+3} - s_n = (-1)^n \ln\left|a^{\frac{n}{3}}\left(c_1 + c_2\cos\frac{2n\pi}{3} + c_3\sin\frac{2n\pi}{3}\right) + \frac{b}{1-a}\right|$$

which is a third order non homogeneous difference equation that can be solved recursively. Let s_0, s_1 and s_2 be given and let

$$f(n) = (-1)^n \ln\left|a^{\frac{n}{3}}\left(c_1 + c_2\cos\frac{2n\pi}{3} + c_3\sin\frac{2n\pi}{3}\right) + \frac{b}{1-a}\right|$$

then

$$s_{n+3} = s_n + f(n)$$

$$\begin{aligned} s_3 &= s_0 + f(0) \\ s_6 &= s_3 + f(3) = s_0 + f(0) + f(3) \\ s_9 &= s_6 + f(6) = s_0 + f(0) + f(3) + f(6) \\ s_{12} &= s_9 + f(9) = s_0 + f(0) + f(3) + f(6) + f(9) \end{aligned}$$

so, for $n = 3m, \qquad m = 1, 2, 3, ...$

$$\begin{aligned} s_n &= s_0 + f(0) + f(3) + f(6) + f(9) + ... + f(n-3) \\ &= s_0 + \sum_{k=0}^{\frac{n-3}{3}} f(3k) \end{aligned}$$

hence,

$$s_{3m} = s_0 + \sum_{k=0}^{m-1} f(3k)$$

but

$$\begin{aligned} f(3k) &= (-1)^{3k} \ln \left| a^{\frac{3k}{3}} \big(c_1 + c_2 \cos \frac{2(3k)\pi}{3} + c_3 \sin \frac{2(3k)\pi}{3}\big) + \frac{b}{1-a} \right| \\ &= (-1)^{3k} \ln \left| a^k \big(c_1 + c_2\big) + \frac{b}{1-a} \right| \end{aligned}$$

thus,

$$s_{3m} = s_0 + \sum_{k=0}^{m-1} (-1)^{3k} \ln \left| a^k \big(c_1 + c_2\big) + \frac{b}{1-a} \right|, \qquad m = 1, 2, 3, ... \tag{3.34}$$

We also have

$$\begin{aligned} s_4 &= s_1 + f(1) \\ s_7 &= s_4 + f(4) = s_1 + f(1) + f(4) \\ s_{10} &= s_7 + f(7) = s_1 + f(1) + f(4) + f(7) \\ s_{13} &= s_{10} + f(10) = s_1 + f(1) + f(4) + f(7) + f(10) \end{aligned}$$

so, for $n = 3m + 1, \; m = 1, 2, 3, ...$

$$\begin{aligned} s_n &= s_1 + f(1) + f(4) + f(7) + f(10) + ... + f(n-3) \\ &= s_1 + \sum_{k=0}^{\frac{n-4}{3}} f(3k+1) \end{aligned}$$

hence

$$s_{3m+1} = s_1 + \sum_{k=0}^{m-1} f(3k+1)$$

but

$$\begin{aligned} f(3k+1) &= (-1)^{3k+1}\ln\left|a^{\frac{3k+1}{3}}\big(c_1 + c_2\cos\frac{2(3k+1)\pi}{3} + c_3\sin\frac{2(3k+1)\pi}{3}\big) + \frac{b}{1-a}\right| \\ &= (-1)^{3k+1}\ln\left|a^{k+\frac{1}{3}}\big(c_1 + c_2\cos\frac{2\pi}{3} + c_3\sin\frac{2\pi}{3}\big) + \frac{b}{1-a}\right| \\ &= (-1)^{3k+1}\ln\left|a^{k+\frac{1}{3}}\big(c_1 + \frac{1}{2}c_2 - \frac{\sqrt{3}}{2}c_3\big) + \frac{b}{1-a}\right| \end{aligned}$$

Thus,

$$s_{3m+1} = s_1 + \sum_{k=0}^{m-1}(-1)^{3k+1}\ln\left|a^{k+\frac{1}{3}}\big(c_1 + \frac{1}{2}c_2 - \frac{\sqrt{3}}{2}c_3\big) + \frac{b}{1-a}\right|, \quad m = 1,2,3,.. \tag{3.35}$$

And finally

$$\begin{aligned} s_5 &= s_2 + f(2) \\ s_8 &= s_5 + f(5) = s_2 + f(2) + f(5) \\ s_{11} &= s_8 + f(8) = s_2 + f(2) + f(5) + f(8) \\ s_{14} &= s_{11} + f(11) = s_2 + f(2) + f(5) + f(8) + f(11) \end{aligned}$$

so, for $n = 3m+2,\ m = 1,2,3,...$

$$\begin{aligned} s_n &= s_2 + f(2) + f(5) + f(8) + f(11) + ... + f(n-3) \\ &= s_2 + \sum_{k=0}^{\frac{n-5}{3}} f(3k+2) \end{aligned}$$

hence,

$$s_{3m+2} = s_2 + \sum_{k=0}^{m-1} f(3k+2)$$

but

$$\begin{aligned} f(3k+2) &= (-1)^{3k+2}\ln\left|a^{\frac{3k+2}{3}}\big(c_1+c_2\cos\frac{2(3k+2)\pi}{3}+c_3\sin\frac{2(3k+2)\pi}{3}\big)+\frac{b}{1-a}\right| \\ &= (-1)^{3k+2}\ln\left|a^{k+\frac{2}{3}}\big(c_1+c_2\cos\frac{4\pi}{3}+c_3\sin\frac{4\pi}{3}\big)+\frac{b}{1-a}\right| \\ &= (-1)^{3k+2}\ln\left|a^{k+\frac{2}{3}}\big(c_1-\frac{1}{2}c_2-\frac{\sqrt{3}}{2}c_3\big)+\frac{b}{1-a}\right| \end{aligned}$$

thus,

$$s_{3m+2} = s_2+\sum_{k=0}^{m-1}(-1)^{3k+2}\ln\left|a^{k+\frac{2}{3}}\big(c_1-\frac{1}{2}c_2-\frac{\sqrt{3}}{2}c_3\big)+\frac{b}{1-a}\right|,\ m=1,2,3,.. \tag{3.36}$$

Now, from (3.34,3.35,3.36), we obtain

$$s_{3m+l} = s_l+\sum_{k=0}^{m-1}(-1)^{3k+l}\ln\left|a^{k+\frac{l}{3}}\big(c_1+c_2\cos\frac{2l\pi}{3}+c_3\sin\frac{2l\pi}{3}\big)+\frac{b}{1-a}\right|,\ l=1,2,3.$$

The canonical coordinate

$$s_{3m+l} = (-1)^{3m+l}\ln|u_{3m+l}|,\qquad l=0,1,2\quad m=1,2,3,...$$

which implies

$$\begin{aligned} u_{3m+l} &= exp\big((-1)^{3m+l}s_{3m+l}\big) \\ &= \exp\left((-1)^{3m+l}s_l+\sum_{k=0}^{m-1}(-1)^{3k+l+3m+l}\ \ln\left|a^{k+\frac{l}{3}}\big(c_1+c_2\cos\frac{2l\pi}{3}+c_3\sin\frac{2l\pi}{3}\big)+\frac{b}{1-a}\right|\right) \\ &= \exp\left((-1)^{3m+l}(-1)^l\ln|u_l|+\sum_{k=0}^{m-1}(-1)^{3k+3m}\ln\left|a^{k+\frac{l}{3}}\big(c_1+c_2\cos\frac{2l\pi}{3}+c_3\sin\frac{2l\pi}{3}\big)+\frac{b}{1-a}\right|\right) \\ &= \exp\left((-1)^{3m+2l}\ln|u_l|+\sum_{k=0}^{m-1}\ln\left|a^{k+\frac{l}{3}}\big(c_1+c_2\cos\frac{2l\pi}{3}+c_3\sin\frac{2l\pi}{3}\big)+\frac{b}{1-a}\right|^{(-1)^{3k+3m}}\right) \\ &= u_l^{(-1)^{3m}}\prod_{k=0}^{m-1}\left(a^{k+\frac{l}{3}}\big(c_1+c_2\cos\frac{2l\pi}{3}+c_3\sin\frac{2l\pi}{3}\big)+\frac{b}{1-a}\right)^{(-1)^{3k+3m}} \end{aligned} \tag{3.37}$$

To verify our computations, we want to show that solution (3.37) verify equation (3.27), which can be written as

$$\frac{u_n}{u_{n+6}} = a + bu_{n+3}u_n$$

The left hand side

$$\begin{aligned}
\frac{u_n}{u_{n+6}} &= \frac{u_{3m+l}}{u_{3(m+2)+l}} \\
&= \frac{u_l^{(-1)^{3m}} \prod_{k=0}^{m-1} \left(a^{k+\frac{l}{3}}\left(c_1 + c_2\cos\frac{2l\pi}{3} + c_3\sin\frac{2l\pi}{3}\right) + \frac{b}{1-a}\right)^{(-1)^{3k+3m}}}{u_l^{(-1)^{3(m+2)}} \prod_{k=0}^{m+1} \left(a^{k+\frac{l}{3}}\left[c_1 + c_2\cos\frac{2l\pi}{3} + c_3\sin\frac{2l\pi}{3}\right] + \frac{b}{1-a}\right)^{(-1)^{3k+3(m+2)}}}
\end{aligned}$$

$$\begin{aligned}
= \; & \frac{u_l^{(-1)^{3m}} \prod_{k=0}^{m-1} \left(a^{k+\frac{l}{3}}\left(c_1 + c_2\cos\frac{2l\pi}{3} + c_3\sin\frac{2l\pi}{3}\right) + \frac{b}{1-a}\right)^{(-1)^{3k+3m}}}{u_l^{(-1)^{3m}} \prod_{k=0}^{m-1} \left(a^{k+\frac{l}{3}}\left[c_1 + c_2\cos\frac{2l\pi}{3} + c_3\sin\frac{2l\pi}{3}\right] + \frac{b}{1-a}\right)^{(-1)^{3k+3m}}} \\
& \cdot \frac{1}{\left(a^{m+\frac{l}{3}}\left[c_1 + c_2\cos\frac{2l\pi}{3} + c_3\sin\frac{2l\pi}{3}\right] + \frac{b}{1-a}\right)^{(-1)^{3m+3m}}} \\
& \cdot \frac{1}{\left(a^{m+1+\frac{l}{3}}\left[c_1 + c_2\cos\frac{2l\pi}{3} + c_3\sin\frac{2l\pi}{3}\right] + \frac{b}{1-a}\right)^{(-1)^{3(m+1)+3m}}}
\end{aligned}$$

$$= \frac{\left(a^{m+1+\frac{l}{3}}\left[c_1 + c_2\cos\frac{2l\pi}{3} + c_3\sin\frac{2l\pi}{3}\right] + \frac{b}{1-a}\right)}{\left(a^{m+\frac{l}{3}}\left[c_1 + c_2\cos\frac{2l\pi}{3} + c_3\sin\frac{2l\pi}{3}\right] + \frac{b}{1-a}\right)}$$

Now, $a + bu_{n+3}u_n$

$$\begin{aligned}
u_{n+3}u_n &= u_{3(m+1)+l}u_{3m+l} \\
&= u_l^{(-1)^{3(m+1)}} \prod_{k=0}^{m} \left(a^{k+\frac{l}{3}} \big(c_1 + c_2 \cos\frac{2l\pi}{3} + c_3 \sin\frac{2l\pi}{3} \big) + \frac{b}{1-a} \right)^{(-1)^{3k+3(m+1)}} \\
&\quad \cdot u_l^{(-1)^{3m}} \prod_{k=0}^{m-1} \left(a^{k+\frac{l}{3}} \big(c_1 + c_2 \cos\frac{2l\pi}{3} + c_3 \sin\frac{2l\pi}{3} \big) + \frac{b}{1-a} \right)^{(-1)^{3k+3m}} \\
&= u_l^{(-1)^{3m+1}} u_l^{(-1)^{3m}} \prod_{k=0}^{m-1} \left(a^{k+\frac{l}{3}} \big(c_1 + c_2 \cos\frac{2l\pi}{3} + c_3 \sin\frac{2l\pi}{3} \big) + \frac{b}{1-a} \right)^{(-1)^{3k+3m+1}} \\
&\quad \left(a^{m+\frac{l}{3}} \big(c_1 + c_2 \cos\frac{2l\pi}{3} + c_3 \sin\frac{2l\pi}{3} \big) + \frac{b}{1-a} \right)^{(-1)^{3m+3m+1}} \\
&\quad \prod_{k=0}^{m-1} \left(a^{k+\frac{l}{3}} \big(c_1 + c_2 \cos\frac{2l\pi}{3} + c_3 \sin\frac{2l\pi}{3} \big) + \frac{b}{1-a} \right)^{(-1)^{3k+3m}} \\
&= \left(a^{m+\frac{l}{3}} \big(c_1 + c_2 \cos\frac{2l\pi}{3} + c_3 \sin\frac{2l\pi}{3} \big) + \frac{b}{1-a} \right)^{-1}
\end{aligned}$$

so

$$\begin{aligned}
a + bu_{n+3}u_n &= a + \frac{b}{\left(a^{m+\frac{l}{3}} \big(c_1 + c_2 \cos\frac{2l\pi}{3} + c_3 \sin\frac{2l\pi}{3} \big) + \frac{b}{1-a} \right)} \\
&= \frac{a^{m+1+\frac{l}{3}} \big(c_1 + c_2 \cos\frac{2l\pi}{3} + c_3 \sin\frac{2l\pi}{3} \big) + \frac{ab}{1-a} + b}{a^{m+\frac{l}{3}} \big(c_1 + c_2 \cos\frac{2l\pi}{3} + c_3 \sin\frac{2l\pi}{3} \big) + \frac{b}{1-a}} \\
&= \frac{a^{m+1+\frac{l}{3}} \big(c_1 + c_2 \cos\frac{2l\pi}{3} + c_3 \sin\frac{2l\pi}{3} \big) + \frac{b}{1-a}}{a^{m+\frac{l}{3}} \big(c_1 + c_2 \cos\frac{2l\pi}{3} + c_3 \sin\frac{2l\pi}{3} \big) + \frac{b}{1-a}}
\end{aligned}$$

Now, to determine the forbidden set from (3.37), $n = 3m + l$, $l = 0, 1, 2$

$$u_n = u_{3m+l} = u_l^{(-1)^{3m+l}} \prod_{k=0}^{m-1} \left(a^{k+\frac{l}{3}} \big(c_1 + c_2 \cos\frac{2l\pi}{3} + c_3 \sin\frac{2l\pi}{3} \big) + \frac{b}{1-a} \right)^{(-1)^{3(k+m)}}$$

let

$$f_l(m,k) = \left(a^{k+\frac{l}{3}} \big(c_1 + c_2 \cos\frac{2l\pi}{3} + c_3 \sin\frac{2l\pi}{3} \big) + \frac{b}{1-a} \right)^{(-1)^{3(k+m)}}$$

then

$$
\begin{aligned}
f_l(m,k) \quad = \quad & \Bigg(\frac{a^{k+\frac{l}{3}}}{3}\Bigg[\frac{1}{u_0u_3}+\frac{1}{u_1u_4}\big(-a^{-\frac{1}{3}}\big)+\frac{1}{u_2u_5}\big(a^{-\frac{2}{3}}\big)-\frac{b}{1-a}\Big(1-a^{-\frac{1}{3}}+a^{-\frac{2}{3}}\Big)\Bigg] \\
& +\frac{a^{k+\frac{l}{3}}}{3}\Bigg[\frac{2}{u_0u_3}+\frac{1}{u_1u_4}\big(-a^{-\frac{1}{3}}\big)+\frac{1}{u_2u_5}\big(-a^{-\frac{2}{3}}\big)-\frac{b}{1-a}\Big(2-a^{-\frac{1}{3}}-a^{-\frac{2}{3}}\Big)\Bigg]\cos\frac{2l\pi}{3} \\
& + \frac{a^{k+\frac{l}{3}}}{\sqrt{3}}\Bigg[\frac{1}{u_1u_4}\big(a^{-\frac{1}{3}}\big)+\frac{1}{u_2u_5}\big(-a^{-\frac{2}{3}}\big)-\frac{b}{1-a}\Big(a^{-\frac{1}{3}}-a^{-\frac{2}{3}}\Big)\Bigg]\sin\frac{2l\pi}{3}) \\
& + \frac{b}{1-a}\Bigg)^{(-1)^{3k+3m}}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \Bigg(\frac{1}{u_0u_3}\Bigg[\frac{a^{k+\frac{l}{3}}}{3}+\frac{2a^{k+\frac{l}{3}}}{3}\cos\frac{2l\pi}{3}\Bigg] \\
& +\frac{1}{u_1u_4}\Bigg[\frac{a^{k+\frac{l}{3}}}{3}\big(-a^{-\frac{1}{3}}\big)+\frac{a^{k+\frac{l}{3}}}{3}\big(-a^{-\frac{1}{3}}\big)\cos\frac{2l\pi}{3}+\frac{a^{k+\frac{l}{3}}}{\sqrt{3}}\big(a^{-\frac{1}{3}}\big)\sin\frac{2l\pi}{3}\Bigg] \\
& +\frac{1}{u_2u_5}\Bigg[\frac{a^{k+\frac{l}{3}}}{3}\big(-a^{\frac{2}{3}}\big)+\frac{a^{k+\frac{l}{3}}}{3}\big(-a^{-\frac{2}{3}}\big)cos\frac{2l\pi}{3}+\frac{a^{k+\frac{l}{3}}}{\sqrt{3}}\big(-a^{-\frac{2}{3}}\big)\sin\frac{2l\pi}{3}\Bigg] \\
& -\frac{b}{1-a}\Bigg[\big(1-a^{-\frac{1}{3}}+a^{-\frac{2}{3}}\big) \\
& \qquad +\big(2-a^{-\frac{1}{3}}-a^{-\frac{2}{3}}\big)\cos\frac{2l\pi}{3} \\
& \qquad +\big(a^{-\frac{1}{3}}-a^{-\frac{2}{3}}\big)\sin\frac{2l\pi}{3}-1\Bigg]\Bigg)^{(-1)^{3k+3m}} \qquad (3.38)
\end{aligned}
$$

$$
\begin{aligned}
= & \Bigg(\frac{u_1u_2u_4u_5(1-a)}{u_0u_1u_2u_3u_4u_5(1-a)} \left[\frac{a^{k+\frac{l}{3}}}{3} + \frac{2a^{k+\frac{l}{3}}}{3}\cos\frac{2l\pi}{3} \right] \\
& + \frac{u_0u_2u_3u_5(1-a)}{u_0u_1u_2u_3u_4u_5(1-a)} \Bigg[\frac{a^{k+\frac{l}{3}}}{3}\big(-a^{-\frac{1}{3}}\big) \\
& \qquad + \frac{a^{k+\frac{l}{3}}}{3}\big(-a^{-\frac{1}{3}}\big)\cos\frac{2l\pi}{3} + \frac{a^{k+\frac{l}{3}}}{\sqrt{3}}\big(a^{-\frac{1}{3}}\big)\sin\frac{2l\pi}{3} \Bigg] \\
& + \frac{u_0u_1u_3u_4(1-a)}{u_0u_1u_2u_3u_4u_5(1-a)} \Bigg[\frac{a^{k+\frac{l}{3}}}{3}\big(-a^{\frac{2}{3}}\big) \\
& \qquad + \frac{a^{k+\frac{l}{3}}}{3}\big(-a^{-\frac{2}{3}}\big)\cos\frac{2l\pi}{3} + \frac{a^{k+\frac{l}{3}}}{\sqrt{3}}\big(-a^{-\frac{2}{3}}\big)\sin\frac{2l\pi}{3} \Bigg] \\
& - \frac{u_0u_1u_2u_3u_4u_5b}{u_0u_1u_2u_3u_4u_5(1-a)} \\
& \qquad \cdot \Bigg[\big(1-a^{-\frac{1}{3}}+a^{-\frac{2}{3}}\big) + \big(2-a^{-\frac{1}{3}}-a^{-\frac{2}{3}}\big)\cos\frac{2l\pi}{3} \\
& \qquad\qquad + \big(a^{-\frac{1}{3}}-a^{-\frac{2}{3}}\big)\sin\frac{2l\pi}{3} - 1 \Bigg] \Bigg)^{(-1)^{3k+3m}}
\end{aligned}
\tag{3.39}
$$

Now, we have the following results

1. if $k+m =$ even number for some m and k, from (3.39) we we have $f_l(m,k)$ is defined for all
$$u_0u_1u_2u_3u_4u_5 \neq 0$$

2. if $k+m =$ odd number for some m and k, from (3.38) we have $f_l(m,k)$ is undefined when

$$
\begin{aligned}
0 = & \Bigg(\frac{1}{u_0u_3} \left[\frac{a^{k+\frac{l}{3}}}{3} + \frac{2a^{k+\frac{l}{3}}}{3}\cos\frac{2l\pi}{3} \right] \\
& + \frac{1}{u_1u_4} \left[\frac{a^{k+\frac{l}{3}}}{3}\big(-a^{-\frac{1}{3}}\big) + \frac{a^{k+\frac{1}{3}}}{3}\big(-a^{-\frac{1}{3}}\big)\cos\frac{2l\pi}{3} + \frac{a^{k+\frac{l}{3}}}{\sqrt{3}}\big(a^{-\frac{1}{3}}\big)\sin\frac{2l\pi}{3} \right] \\
& + \frac{1}{u_2u_5} \left[\frac{a^{k+\frac{l}{3}}}{3}\big(-a^{\frac{2}{3}}\big) + \frac{a^{k+\frac{l}{3}}}{3}\big(-a^{-\frac{2}{3}}\big)\cos\frac{2l\pi}{3} + \frac{a^{k+\frac{l}{3}}}{\sqrt{3}}\big(-a^{-\frac{2}{3}}\big)\sin\frac{2l\pi}{3} \right] \\
& - \frac{b}{1-a} \left[\big(1-a^{-\frac{1}{3}}+a^{-\frac{2}{3}}\big) + \big(2-a^{-\frac{1}{3}}-a^{-\frac{2}{3}}\big)\cos\frac{2l\pi}{3} + \big(a^{-\frac{1}{3}}-a^{-\frac{2}{3}}\big)\sin\frac{2l\pi}{3} - 1 \right] \Bigg) \\
= & \frac{1}{u_0u_3}\xi^l_{1k} + \frac{1}{u_1u_4}\xi^l_{2k} + \frac{1}{u_2u_5}\xi^l_{3k} - \frac{b}{1-a}\xi^l_4
\end{aligned}
$$

where, for $l = 0, 1, 2$

$$\begin{aligned}
\xi_{1k}^l &= \left[\frac{a^{k+\frac{l}{3}}}{3} + \frac{2a^{k+\frac{l}{3}}}{3}\cos\frac{2l\pi}{3}\right] \\
\xi_{2k}^l &= \left[\frac{a^{k+\frac{l}{3}}}{3}\big(-a^{-\frac{1}{3}}\big) + \frac{a^{k+\frac{l}{3}}}{3}\big(-a^{-\frac{1}{3}}\big)\cos\frac{2l\pi}{3} + \frac{a^{k+\frac{l}{3}}}{\sqrt{3}}\big(a^{-\frac{1}{3}}\big)\sin\frac{2l\pi}{3}\right] \\
\xi_{3k}^l &= \left[\frac{a^{k+\frac{l}{3}}}{3}\big(-a^{\frac{2}{3}}\big) + \frac{a^{k+\frac{l}{3}}}{3}\big(-a^{-\frac{2}{3}}\big)\cos\frac{2l\pi}{3} + \frac{a^{k+\frac{l}{3}}}{\sqrt{3}}\big(-a^{-\frac{2}{3}}\big)\sin\frac{2l\pi}{3}\right] \\
\xi_4^l &= \left[\big(1 - a^{-\frac{1}{3}} + a^{-\frac{2}{3}}\big) + \big(2 - a^{-\frac{1}{3}} - a^{-\frac{2}{3}}\big)\cos\frac{2l\pi}{3} + \big(a^{-\frac{1}{3}} - a^{-\frac{2}{3}}\big)\sin\frac{2l\pi}{3} - 1\right]
\end{aligned}$$

Theorem 3.4.1. *Let $u_0, u_1, u_2, u_3, u_4, u_5 \in \mathbb{R}$ such that $u_0u_1u_2u_3u_4u_5 \neq 0$ and let $a \neq 1$. Then the forbidden set $\mathcal{F}$ of the difference equation (3.27) is given by $\mathcal{F} = \mathcal{F}^0 \bigcup \mathcal{F}^1 \bigcup \mathcal{F}^2$, where*

$$\mathcal{F}^l = \bigcup_{k=1}^{\infty}\left\{\frac{1}{u_0u_3}\xi_{1k}^l \;+\; \frac{1}{u_1u_4}\xi_{2k}^l \;+\; \frac{1}{u_2u_5}\xi_{3k}^l \;-\; \frac{b}{1-a}\xi_4^l = 0\right\}$$

3.4.2 The Case $a = 1$

We consider the case $a = 1$. Equation (3.32) becomes

$$z_{n+3} - z_n = b \tag{3.40}$$

the characteristic roots of the homogeneous equation, $z_{n+3} - z_n = 0$, are the roots of the characteristic equation

$$\lambda^3 - 1 = 0$$

so

$$(\lambda - 1)(\lambda^2 + \lambda + 1) = 0$$

and so

$$\lambda_1 = 1, \quad \lambda_{2,3} = \frac{-1 \pm \sqrt{3}i}{2}.$$

Hence, the homogeneous solution z_h is

$$z_h = c_1 + c_2\cos\frac{2n\pi}{3} + c_3\sin\frac{2n\pi}{3},$$

let the particular solution be

$$z_p = nc,$$

substitute into (3.40), we obtain

$$(n+3)c - nc = b$$

so

$$c = \frac{b}{3}$$

then

$$z_p = \frac{b}{3}n$$

thus, the solution of equation (3.40) is

$$\begin{aligned} z_n &= z_h + z_p \\ &= c_1 + c_2 \cos\frac{2n\pi}{3} + c_3 \sin\frac{2n\pi}{3} + \frac{n}{3}b. \end{aligned}$$

Given initial values z_0, z_1, z_2, the constants c_1, c_2, c_3 satisfy the following system of equations

$$\begin{aligned} z_0 &= c_1 + c_2 \\ z_1 - \frac{b}{3} &= c_1 - \frac{1}{2}c_2 + \frac{\sqrt{3}}{2}c_3 \\ z_1 - \frac{2b}{3} &= c_1 - \frac{1}{2}c_2 - \frac{\sqrt{3}}{2}c_3 \end{aligned}$$

solving the last system for c_1, c_2, c_3, we get

$$\begin{aligned} c_1 &= \frac{1}{3}(z_0 + z_1 + z_2 - b) = \frac{1}{3}\Big(\frac{1}{u_0u_3} + \frac{1}{u_1u_4} + \frac{1}{u_2u_5} - b\Big) \\ c_2 &= \frac{1}{3}(2z_0 - z_1 - z_2 + b) = \frac{1}{3}\Big(2\frac{1}{u_0u_3} - \frac{1}{u_1u_4} - \frac{1}{u_2u_5} + b\Big) \\ c_3 &= \frac{1}{\sqrt{3}}(z_1 - z_2 + \frac{b}{3}) = \frac{1}{\sqrt{3}}\Big(\frac{1}{u_1u_4} - \frac{1}{u_2u_5} + \frac{b}{3}\Big) \end{aligned}$$

The invariant v_n is given by

$$\begin{aligned} v_n &= \frac{1}{z_n} \\ &= \frac{1}{c_1 + c_2\cos\frac{2n\pi}{3} + c_3\sin\frac{2n\pi}{3} + \frac{n}{3}b} = u_nu_{n+3} \end{aligned}$$

so

$$u_{n+3} = \frac{1}{u_n\left(c_1 + c_2\cos\frac{2n\pi}{3} + c_3\sin\frac{2n\pi}{3} + \frac{n}{3}b\right)}.$$

To solve the last equation we use the canonical coordinate,

$$s_n = \int \frac{du_n}{(-1)^n u_n}$$

$$= (-1)^n \ln|u_n|$$

so

$$\begin{aligned}
s_{n+3} - s_n &= (-1)^{n+3}\ln|u_{n+3}| - (-1)^n \ln|u_n| \\
&= (-1)^{n+1}[\ln|u_{n+3}| + \ln|u_n|] \\
&= (-1)^{n+1}\ln|u_{n+3}u_n| \\
&= (-1)^{n+1}\ln|v_n| \\
&= (-1)^{n+1}\ln\left|\frac{1}{c_1 + c_2\cos\frac{2n\pi}{3} + c_3\sin\frac{2n\pi}{3} + \frac{n}{3}b}\right| \\
&= (-1)^{n+2}\ln\left|c_1 + c_2\cos\frac{2n\pi}{3} + c_3\sin\frac{2n\pi}{3} + \frac{n}{3}b\right|
\end{aligned}$$

and so

$$s_{n+3} - s_n = (-1)^n \ln\left|c_1 + c_2\cos\frac{2n\pi}{3} + c_3\sin\frac{2n\pi}{3} + \frac{n}{3}b\right|$$

let

$$f(n) = (-1)^n \ln\left|c_1 + c_2\cos\frac{2n\pi}{3} + c_3\sin\frac{2n\pi}{3} + \frac{n}{3}b\right|,$$

then

$$s_{n+3} - s_n = f(n)$$

which is a third order non homogeneous difference equation that can be solved recursively as in the previous case when $a \neq 1$. Let s_0, s_1 and s_2 be given, then

$$s_{3m+l} = s_l + \sum_{k=0}^{m-1}(-1)^{3k+l}\ln\left|c_1 + c_2\cos\frac{2l\pi}{3} + c_3\sin\frac{2l\pi}{3} + \frac{l}{3}b\right|,\ l = 0,1,2,\ \ m = 1,2,3,...$$

The canonical coordinate

$$s_{3m+l} = (-1)^{3m+l}\ln|u_{3m+l}|, \qquad l = 0,1,2\ \ m = 1,2,3,...$$

which implies

$$\begin{aligned} u_{3m+l} &= \exp\left((-1)^{3m+l}s_{3m+l}\right) \\ &= u_l^{(-1)^{3m}} \prod_{k=0}^{m-1}\left(c_1 + c_2\cos\frac{2l\pi}{3} + c_3\sin\frac{2l\pi}{3} + \frac{l}{3}b\right)^{(-1)^{3k+3m}} \end{aligned} \quad (3.41)$$

Now, to determine the forbidden set. From (3.41), $n = 3m + l, \quad l = 0, 1, 2$

$$u_n = u_{3m+l} = u_l^{(-1)^{3m}} \prod_{k=0}^{m-1}\left(c_1 + c_2\cos\frac{2l\pi}{3} + c_3\sin\frac{2l\pi}{3} + \frac{l}{3}b\right)^{(-1)^{3k+3m}}$$

let

$$f_l(m,k) = \left(c_1 + c_2\cos\frac{2l\pi}{3} + c_3\sin\frac{2l\pi}{3} + \frac{l}{3}b\right)^{(-1)^{3k+3m}}$$

then

$$\begin{aligned} f_l(m,k) = &\left(\frac{1}{u_0u_3}\left[\frac{1}{3} + \frac{2}{3}\cos\frac{2l\pi}{3}\right] + \frac{1}{u_1u_4}\left[\frac{1}{3} - \frac{1}{3}\cos\frac{2l\pi}{3} + \frac{1}{\sqrt{3}}\sin\frac{2l\pi}{3}\right]\right. \\ &+\frac{1}{u_2u_5}\left[\frac{1}{3} - \frac{1}{3}\cos\frac{2l\pi}{3} - \frac{1}{\sqrt{3}}\sin\frac{2l\pi}{3}\right] \\ &\left.-\frac{b}{3}\left[1 - \cos\frac{2l\pi}{3} - \frac{1}{\sqrt{3}}\sin\frac{2l\pi}{3} - l\right]\right)^{(-1)^{3k+3m}} \end{aligned} \quad (3.42)$$

$$\begin{aligned} = &\left(\frac{3u_1u_2u_4u_5}{3u_0u_1u_2u_3u_4u_5}\left[\frac{1}{3} + \frac{2}{3}\cos\frac{2l\pi}{3}\right]\right. \\ &+ \frac{3u_0u_2u_3u_5}{3u_0u_1u_2u_3u_4u_5}\left[\frac{1}{3} - \frac{1}{3}\cos\frac{2l\pi}{3} + \frac{1}{\sqrt{3}}\sin\frac{2l\pi}{3}\right] \\ &+\frac{3u_0u_1u_3u_4}{3u_0u_1u_2u_3u_4u_5}\left[\frac{1}{3} - \frac{1}{3}\cos\frac{2l\pi}{3} - \frac{1}{\sqrt{3}}\sin\frac{2l\pi}{3}\right] \\ &\left.-\frac{bu_0u_1u_2u_3u_4u_5}{3u_0u_1u_2u_3u_4u_5u_6}\left[1 - \cos\frac{2l\pi}{3} - \frac{1}{\sqrt{3}}\sin\frac{2l\pi}{3} - l\right]\right)^{(-1)^{3(k+m)}} \end{aligned} \quad (3.43)$$

Now,

1. if $k+m$ = even number for some m and k, from (3.43) we have $f_l(m,k)$ is defined where

$$u_0u_1u_2u_3u_4u_5 \neq 0$$

2. if $k+m =$ odd number for some m and k, from (3.42) we have $f_l(m,k)$ is undefined when

$$\begin{aligned} 0 &= \left(\frac{1}{u_0u_3}\left[\frac{1}{3}+\frac{2}{3}\cos\frac{2l\pi}{3}\right]+\frac{1}{u_1u_4}\left[\frac{1}{3}-\frac{1}{3}\cos\frac{2l\pi}{3}+\frac{1}{\sqrt{3}}\sin\frac{2l\pi}{3}\right]\right.\\ &\qquad +\frac{1}{u_2u_5}\left[\frac{1}{3}-\frac{1}{3}\cos\frac{2l\pi}{3}-\frac{1}{\sqrt{3}}\sin\frac{2l\pi}{3}\right]\\ &\qquad \left.-\frac{b}{3}\left[1-\cos\frac{2l\pi}{3}-\frac{1}{\sqrt{3}}\sin\frac{2l\pi}{3}-l\right]\right)\\ &= \frac{1}{u_0u_3}\xi_1^l + \frac{1}{u_1u_4}\xi_2^l + \frac{1}{u_2u_5}\xi_3^l - \frac{b}{3}\xi_4^l \end{aligned}$$

where

$$\begin{aligned} \xi_1^l &= \left[\frac{1}{3}+\frac{2}{3}\cos\frac{2l\pi}{3}\right]\\ \xi_2^l &= \left[\frac{1}{3}-\frac{1}{3}\cos\frac{2l\pi}{3}+\frac{1}{\sqrt{3}}\sin\frac{2l\pi}{3}\right]\\ \xi_3^l &= \left[\frac{1}{3}-\frac{1}{3}\cos\frac{2l\pi}{3}-\frac{1}{\sqrt{3}}\sin\frac{2l\pi}{3}\right]\\ \xi_4^l &= \left[1-\cos\frac{2l\pi}{3}-\frac{1}{\sqrt{3}}\sin\frac{2l\pi}{3}-l\right]. \end{aligned}$$

We get the following theorem

Theorem 3.4.2. *Let $u_0, u_1, u_2, u_3, u_4, u_5 \in \mathbb{R}$ such that $u_0u_1u_2u_3u_4u_5 \neq 0$ and let $a = 1$. Then the forbidden set $\mathcal{F}$ of the difference equation (3.27) is given by $\mathcal{F} = \mathcal{F}^0 \bigcup \mathcal{F}^1 \bigcup \mathcal{F}^2$ where*

$$\mathcal{F}^l = \left\{\frac{1}{u_0u_3}\xi_1^l + \frac{1}{u_1u_4}\xi_2^l + \frac{1}{u_2u_5}\xi_3^l - \frac{b}{3}\xi_4^l = 0\right\}$$

CONCLUSION

By the method of symmetry, we have solved the difference equation

$$u_{n+2(i+1)} = \frac{u_n}{a + bu_{n+i+1}u_n},$$

when i is Even, that is an open problem proposed in [3]. We have determined the forbidden set of this difference equation. We also have considered the special case $i = 0$ and the special case $i = 2$.

BIBLIOGRAPHY

[1] R. Abo-Zeid. *Forbidden Sets and Stability in Some Rational Difference Equations.* Journal of Difference Equations and Applications, 24(2):220–239, 2018.

[2] R. Azizi. *On The Global Behaviour of Some Difference Equations.* PhD thesis, Faculty of science of Bizerte, 2013.

[3] F. Balibrea and A. Cascales. On forbidden sets. *Journal of Difference Equations and Applications*, 21(10):974–996, 2015.

[4] S. Elaydi. *An Introduction to Difference Equations.* Springer, 2005.

[5] M. Folly-Gbetoula. *Symmetry, reductions and exact solutions of the difference equation.* Journal of Difference Equations and Applications, 23(6):1017–1024, 2017.

[6] M. Folly-Gbetoula. *Symmetries and Conservation Laws of Difference and Iterative Equations.* PhD thesis, University of the Witwatersrand, 2015.

[7] E. A. Grove and G. Ladas *Periodicities in nonlinear difference equations*, vol 4. Chapman & Hall/CRC, London, 2005.

[8] P. E Hydon. *Symmetries and First Integrals of Ordinary Difference Equations.* In Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, volume 456, pages 2835–2855. The Royal Society, 2000.

[9] P. E. Hydon. *Symmetry Methods for Differential Equations: a Beginner's Guide.* Cambridge University Press, 2000.

[10] P. E. Hydon. *Difference Equations by Differential Equation Methods.* Cambridge-University Press, Cambridge, 2014.

[11] J. Rabago. *Forbidden set of the rational difference equation* $x_{n+1} = \frac{x_n x_{nk}}{(a x_{n-k+1} + x_n x_{n-k+1} x_{nk})}$. arXiv preprint arXiv:1604.05949 (2016).

[12] H. Sedaghat. *Global Behaviours of Rational Difference Equations of Orders Two and Three With Quadratic Terms.* Journal of Difference Equations and Applications, 15(3):215–224, 2009.

[13] A. Walter. *Partial Differential Equations, New York, NY, USA: John Wiley & Sons, 1992.* Laurie Rosatone.

[14] W. Yaseen. *Using Symmetries To Solve Some Difference Equations.* M. Sc. thesis, Birzeit University, 2018.